

An elliptic constant in lower bounds for gradient descent

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Abstract

We minimize a lower bound for gradient descent obtained by testing predetermined non-negative schedules on scalar quadratics and on Huber functions whose iterates remain in an affine region. For N steps, its minimum E_N satisfies $N^2 E_N \rightarrow C$, where $0.304194775980 < C < 0.304194775981$. The constant is specified by a unique root of an equation in complete elliptic integrals. The optimizing polynomial at each finite horizon is unique and has only real simple roots. After rescaling, these polynomials converge to an entire function whose differential equation gives the constant formula. Chebyshev composition yields a finite bound, which also applies to predetermined real schedules in every dimension.

1 The lower bound and its asymptotic minimum

For a nonnegative schedule $h = (h_1, \dots, h_N)$, put

$$\mathcal{B}_N(h) = \max \left\{ \frac{1}{2} \max_{0 \leq \lambda \leq 1} \lambda \prod_{i=1}^N (1 - h_i \lambda)^2, \frac{1}{2(1 + 2 \sum_{i=1}^N h_i)} \right\}, \quad E_N = \inf_{h \in [0, \infty)^N} \mathcal{B}_N(h). \quad (1)$$

The first term is the worst terminal value on the quadratics $f_\lambda(x) = \lambda x^2/2$, starting at $x_0 = 1$. For the second, write $S = \sum_i h_i$ and take

$$t = \frac{1}{1 + 2S}, \quad f_t(x) = \begin{cases} x^2/2, & |x| \leq t, \\ t|x| - t^2/2, & |x| \geq t. \end{cases}$$

Every iterate satisfies $x_j = 1 - t \sum_{i \leq j} h_i \geq 1 - tS \geq t$, so all updates use the affine part of f_t . The terminal value is $t - (S + 1/2)t^2 = 1/[2(1 + 2S)]$.

Drori and Teboulle [2, Theorem 3.2] use the quadratic and affine-quadratic tests to obtain their lower bound for constant steps. Equation (1) uses these tests for variable steps and includes all quadratic curvatures in $[0, 1]$.

For $0 \leq q < 1$, we use the modulus convention for the complete elliptic integrals:

$$K(q) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - q^2 \sin^2 t}}, \quad E(q) = \int_0^{\pi/2} \sqrt{1 - q^2 \sin^2 t} dt, \quad v(q) = 2E(q) - K(q). \quad (2)$$

Theorem 1.1. *The function v has a unique zero $q_0 \in (0, 1)$. The equation*

$$\frac{K(q)}{v(q)} - 1 + 2q^2 - v(q)^2 = 0 \quad (3)$$

has a unique solution $q_* \in (0, q_0)$. With

$$U = \frac{K(q_*)}{v(q_*)}, \quad C = \frac{U^2}{8}, \quad (4)$$

one has

$$\lim_{N \rightarrow \infty} N^2 E_N = C, \quad E_N \geq \frac{C}{(N + 1/2)^2 + 4C/3} \quad (N \geq 1). \quad (5)$$

Moreover,

$$0.58413072912712 < q_* < 0.58413072912713, \quad 0.304194775980 < C < 0.304194775981. \quad (6)$$

To state the gradient-descent consequence, let $\mathcal{F}_d$ consist of the convex differentiable functions on $\mathbb{R}^d$ with 1-Lipschitz gradient and an attained minimum. For $h \in \mathbb{R}^N$, define

$$\begin{aligned} x_j &= x_{j-1} - h_j \nabla f(x_{j-1}), \quad 1 \leq j \leq N, \\ W_{N,d}(h) &= \sup_{\substack{f \in \mathcal{F}_d, \ x_* \in \arg \min f \\ \|x_0 - x_*\| \leq 1}} (f(x_N) - f(x_*)), \\ V_{N,d}^+ &= \inf_{h \in [0, \infty)^N} W_{N,d}(h), \quad V_{N,d}^{\mathbb{R}} = \inf_{h \in \mathbb{R}^N} W_{N,d}(h). \end{aligned} \quad (7)$$

The schedule is chosen before the objective and the initial point.

Corollary 1.2. For $N, d \geq 1$ and $\diamond \in \{+, \mathbb{R}\}$,

$$V_{N,d}^{\diamond} \geq E_N \geq \frac{C}{(N + 1/2)^2 + 4C/3} > \frac{c}{(N + 1/2)^2 + 4c/3}, \quad c = 0.304194775980. \quad (8)$$

Consequently every sequence $d_N \geq 1$ satisfies $\liminf_{N \rightarrow \infty} N^2 V_{N,d_N}^{\diamond} \geq C$.

Sections 2–5 prove the exact limit in Theorem 1.1; its finite bound follows from the composition inequality in Section 3. Section 6 proves convergence of the optimizing polynomials and Corollary 1.2, and Section 7 verifies (6).

2 The finite optimizing polynomial

Fix $N \geq 1$ and write $M = 2N + 1$. For any odd polynomial P , put

$$S(P) = -P'''(0)/6, \quad \|P\| = \max_{[-1,1]} |P|.$$

Thus $P(x) = x - S(P)x^3 + O(x^5)$ when $P'(0) = 1$. The schedule polynomial

$$P_h(x) = x \prod_{i=1}^N (1 - h_i x^2) \quad (9)$$

satisfies $S(P_h) = \sum_i h_i$, and its squared norm divided by two is the quadratic term in (1).

Lemma 2.1 (Odd interpolation). *Let $0 < z_1 < \dots < z_N \leq 1$ and define*

$$t = \sum_{j=1}^N z_j^{-2}, \quad b_j = z_j^{-3} \prod_{k \neq j} \frac{z_k^2}{|z_j^2 - z_k^2|}, \quad A = \sum_{j=1}^N b_j.$$

Every odd polynomial of degree at most M with $P'(0) = 1$ satisfies

$$S(P) = t + \sum_{j=1}^N (-1)^j b_j P(z_j) \leq t + A \|P\|. \quad (10)$$

The least possible norm of such a polynomial is $1/M$, uniquely attained by C_M/M , where $C_M = (-1)^N T_M$ and T_M is the Chebyshev polynomial.

Proof. The polynomials

$$H_0(x) = x \prod_{j=1}^N (1 - x^2/z_j^2), \quad H_j(x) = \frac{x^3}{z_j^3} \prod_{k \neq j} \frac{x^2 - z_k^2}{z_j^2 - z_k^2}$$

have the interpolation identity $P = H_0 + \sum_j P(z_j) H_j$. Indeed, both sides have the same derivative at zero and the same values at the N positive nodes, which determine an odd polynomial of degree at most $2N + 1$. Their values of S are $S(H_0) = t$ and $S(H_j) = (-1)^j b_j$, proving (10).

For the norm assertion, take the $N + 1$ positive Chebyshev extrema $x_j = \sin((2j + 1)\pi/(2M))$, $0 \leq j \leq N$, including $x_N = 1$. The odd cardinal polynomials

$$L_j(x) = \frac{x}{x_j} \prod_{k \neq j} \frac{x^2 - x_k^2}{x_j^2 - x_k^2}$$

have derivative weights $L'_j(0)$ of signs $(-1)^j$. Since $C_M(x_j) = (-1)^j$ and $C'_M(0) = M$, interpolation of C_M gives $\sum_j |L'_j(0)| = M$. Hence $1 = P'(0) \leq M \|P\|$. Equality forces $P(x_j) = (-1)^j/M$ at every node, so $P = C_M/M$. $\square$

For $r \geq 1/M$, define

$$\Gamma_N(r) = \max\{S(P) : P \text{ odd, } \deg P \leq M, P'(0) = 1, \|P\| \leq r\}. \quad (11)$$

For a feasible polynomial P , a contact is a point $x \in [-1, 1]$ where $|P(x)| = r$.

Lemma 2.2. *The function Γ_N is continuous and strictly increasing, and $\Gamma_N(1/M) = (M^2 - 1)/6$. For every $r > 1/M$ its maximizing polynomial is unique and has norm r . It has contacts*

$$P(z_j) = (-1)^j r, \quad 0 < z_1 < \dots < z_N = 1. \quad (12)$$

It has N distinct positive zeros, all in $(0, 1)$. Its positive critical points are a point $a \in (0, z_1)$ and the $N - 1$ interior contacts; all its critical points are real and simple, and $0 < P(a) < r$.

Proof. The feasible set in (11) is compact in a finite-dimensional polynomial space. If the allowed radius increases, adding a sufficiently small negative multiple of x^3 increases $S(P)$ while remaining feasible. Thus Γ_N is strictly increasing. Compactness gives upper semicontinuity. For left

continuity at $r > 1/M$, mix a maximizing polynomial with C_M/M , whose norm is strictly smaller. Compactness and the uniqueness in Lemma 2.1 also give continuity at $1/M$. The expansion

$$C_M(x) = \sin(M \arcsin x) = Mx - \frac{M(M^2 - 1)}{6}x^3 + O(x^5)$$

gives the stated endpoint value. A maximizer at $r > 1/M$ must have norm r , since otherwise a small negative cubic perturbation increases its objective.

The positive contact set $\{x \in (0, 1] : |P(x)| = r\}$ is finite. Group its points into successive blocks of equal sign, and let L be the number of blocks. Since P is positive immediately to the right of zero, counting positive zeros gives $L \leq N$ if the first contact is negative, and $L \leq N + 1$ if it is positive.

Suppose first that the initial contact is positive and $L \leq N$. Choose a point ξ_j between each pair of successive sign blocks. Then

$$R(x) = -x^3 \prod_j (1 - x^2/\xi_j^2)$$

has degree at most M , cubic coefficient -1 , and sign opposite to P at every contact. If instead the initial contact is negative and $L \leq N - 1$, the same construction works after adding one root between zero and the first contact. For small $\varepsilon > 0$, $P + \varepsilon R$ has norm less than r and larger S . To justify the norm statement, choose neighborhoods of the finite contact set on which the signs remain strictly opposed. On their compact complement, $|P|$ has a positive gap from r . Uniformly small ε works on both sets. This contradicts maximality.

The remaining positive-first case, $L = N + 1$, gives $M + 1$ alternating contacts on $[-1, 1]$ by odd reflection. Interpolation at the $M + 1$ extrema of T_M bounds the absolute value of the x^M coefficient of any polynomial Q of degree at most M by $2^{M-1}\|Q\|$. Equality holds only for a scalar multiple of T_M : the coefficient weights alternate in sign, and their absolute values sum to 2^{M-1} by interpolation of T_M . Let R be the scalar multiple of T_M with the same leading coefficient as P . The bound gives $\|R\| \leq r$. If $\|R\| < r$, the alternating contacts of P would give at least M zeros of $P - R$, whose degree is at most $M - 1$. Thus $\|R\| = r$, and the equality case makes P a scalar multiple of T_M . Its derivative at zero then forces $r = 1/M$, a contradiction.

There are consequently exactly N alternating positive contact blocks, starting negative. Select one contact from each. The identity (10) is an equality at these nodes. Any other maximizer must have $P(z_j) = (-1)^j r$ at each selected node to attain equality in every summand. These values and the derivative at zero determine the polynomial by odd interpolation, proving uniqueness.

There is one positive zero before the first negative contact and one between every pair of successive contacts. These are N distinct zeros in $(0, 1)$. Their negatives and the origin exhaust degree M . There is a positive critical point before the first contact, and every interior contact is critical. Since the even polynomial P' has at most N distinct positive zeros, the last contact must be at one. The first critical point and the $N - 1$ interior contacts, together with their negatives, exhaust the degree of P' . They are therefore simple, and there are no further contacts or critical points. The initial positive maximum is strictly below r because the first contact is negative. $\square$

Proposition 2.3. *For every $N \geq 1$, the infimum E_N is attained by a unique polynomial P_N of the form (9). If $r_N = \|P_N\|$ and $S_N = S(P_N)$, then*

$$\frac{1}{M} < r_N \leq \frac{1}{2}, \quad E_N = \frac{r_N^2}{2}, \quad 1 + 2S_N = r_N^{-2}. \quad (13)$$

The roots and critical points of P_N have the structure in Lemma 2.2. Its positive roots determine a unique optimizing step multiset, all of whose entries exceed one.

Proof. Among odd polynomials of degree at most M with $P'(0) = 1$, the constraints $\|P\| \leq r$ and $1 + 2S(P) \geq r^{-2}$ are feasible exactly when

$$\Gamma_N(r) \geq \frac{r^{-2} - 1}{2}. \quad (14)$$

At $r = 1/M$, the left side is $(M^2 - 1)/6$, strictly below the right side. At $r = 1/2$, the polynomial $P(x) = x - (3/2)x^3$ has norm $1/2$ and $1 + 2S(P) = 4$. Continuity and strict monotonicity give a unique crossing $r_N \in (1/M, 1/2]$.

By Lemma 2.2, the maximizing polynomial at this radius has N distinct positive roots $\zeta_i \in (0, 1)$. It is $P_N(x) = x \prod_i (1 - x^2/\zeta_i^2)$ and thus corresponds to the positive steps $h_i = \zeta_i^{-2} > 1$. Both terms of its quadratic–Huber bound equal $r_N^2/2$. Conversely, the polynomial of any schedule with bound $\mathcal{B}_N(h) \leq r^2/2$ satisfies (14), so $r \geq r_N$. Equality forces $S(P) = \Gamma_N(r_N)$ and hence fixes the polynomial. This proves (13) and uniqueness. $\square$

3 Chebyshev composition and existence of the limit

Lemma 3.1. *Let P be an odd polynomial of degree $M = 2n + 1$, with derivative one at zero, norm r , $S(P) = S > 0$, and all roots real, simple, and in $(-1, 1)$. Then*

$$\limsup_{N \rightarrow \infty} N^2 E_N \leq \frac{M^2}{8} \max \left\{ r^2, \frac{1}{2S + 1/3} \right\}. \quad (15)$$

Proof. For odd $k \geq 1$, set $C_k = (-1)^{(k-1)/2} T_k$ and $Q(x) = P(C_k(x))/k$. Then

$$Q'(0) = 1, \quad \|Q\| = r/k, \quad S(Q) = k^2(S + 1/6) - 1/6. \quad (16)$$

Every root of P has k distinct preimages in $(-1, 1)$ under C_k . None is a critical value of C_k , whose interior critical values are ± 1 . Thus Q has only real simple roots in the interval and represents a positive schedule of length $(kM - 1)/2$. Its quadratic–Huber bound is

$$\max \left\{ \frac{r^2}{2k^2}, \frac{1}{2[k^2(2S + 1/3) + 2/3]} \right\}.$$

For an arbitrary large N , choose the largest odd k with $kM \leq 2N + 1$, and append zero steps. Since $k \sim 2N/M$, (15) follows. $\square$

Proposition 3.2. *The limit $C = \lim_{N \rightarrow \infty} N^2 E_N$ exists, with $1/8 \leq C \leq 3/8$. For every fixed $n \geq 1$,*

$$C \leq A_n := \frac{(2n + 1)^2}{8(2S_n + 1/3)} = \frac{(2n + 1)^2 r_n^2}{8(1 - 2r_n^2/3)}. \quad (17)$$

Proof. The norm bound in Lemma 2.1 and the schedule represented by C_M/M give

$$\frac{1}{2M^2} \leq E_N \leq \frac{3}{2(M^2 + 2)}, \quad M = 2N + 1. \quad (18)$$

In particular $r_N = O(N^{-1})$. Apply Lemma 3.1 to the finite optimizer P_n . Since $1 + 2S_n = r_n^{-2}$, the second term of the maximum dominates, and $\limsup N^2 E_N \leq A_n$ for every n . Moreover

$$\frac{A_n}{n^2 E_n} = \frac{[(2n + 1)/(2n)]^2}{1 - 2r_n^2/3} \rightarrow 1.$$

Choosing a subsequence along which $n^2 E_n$ tends to its lower limit forces its upper limit to be no larger. The bounds on C follow from (18).

Taking $n = N$ in (17) gives

$$8C(1 - 2r_N^2/3) \leq (2N + 1)^2 r_N^2.$$

Rearranging and using $E_N = r_N^2/2$ proves the finite bound in Theorem 1.1:

$$E_N \geq \frac{4C}{(2N + 1)^2 + 16C/3} = \frac{C}{(N + 1/2)^2 + 4C/3}. \quad \square$$

4 The limiting differential equation

Bogatyrev [1, §1] describes extremal polynomials by algebraic curves determined by the odd-order zeros of $P^2 - 1$. For the present polynomials, removing the double contact zeros from $r_N^2 - P_N^2$ leaves an even quartic.

For the optimizer P_N , write a_N for its first positive critical point, $z_1, \dots, z_{N-1}$ for its interior contacts, and γ_N for its leading coefficient.

Lemma 4.1. *There is a monic even quartic $D_N(x) = x^4 + A_N^f x^2 + B_N^f$ such that*

$$r_N^2 - P_N(x)^2 = \gamma_N^2 (1 - x^2) D_N(x) \prod_{j=1}^{N-1} (x^2 - z_j^2)^2, \quad (19)$$

$$(1 - x^2) D_N(x) P_N'(x)^2 = M^2 (x^2 - a_N^2)^2 (r_N^2 - P_N(x)^2), \quad (20)$$

$$B_N^f = M^2 r_N^2 a_N^4, \quad A_N^f = B_N^f (4S_N - 2a_N^{-2}). \quad (21)$$

Proof. The zeros of $r_N^2 - P_N^2$ at ± 1 are simple, and those at the interior contacts $\pm z_j$ are double. Removing these factors leaves a monic even quartic, proving (19) by comparison of leading coefficients.

The derivative factorization is

$$P_N'(x) = M \gamma_N (x^2 - a_N^2) \prod_{j=1}^{N-1} (x^2 - z_j^2).$$

Canceling in (19) gives (20). The constant and quadratic coefficients in its expansion at zero give

$$B_N^f = M^2 r_N^2 a_N^4, \quad \frac{A_N^f}{B_N^f} = 6S_N + 1 - r_N^{-2} - 2a_N^{-2}.$$

Equation (13) reduces this to (21). $\square$

Set

$$U_N = M r_N, \quad F_N(z) = M P_N(z/M), \quad \alpha_N = M a_N, \quad U = \sqrt{8C}. \quad (22)$$

Proposition 3.2 gives $U_N \rightarrow U$ and $1 \leq U \leq \sqrt{3}$.

Lemma 4.2. *The family (F_N) is locally bounded on $\mathbb{C}$. Every locally uniform limit F is odd, real on $\mathbb{R}$, and satisfies*

$$|F(x + iy)| \leq U e^{|y|}, \quad F'(0) = 1, \quad F'''(0) = -3/U^2. \quad (23)$$

The sequence (α_N) is bounded above and bounded away from zero.

Proof. If P has degree at most M and norm at most r , the function $w^{-M}P((w + w^{-1})/2)$ is analytic for $|w| > 1$, including its removable value at infinity, and has boundary modulus at most r . The maximum principle gives the usual Joukowski bound. For z/M near zero it reads

$$|MP(z/M)| \leq Mr \exp\{M|\operatorname{Im} \arcsin(z/M)|\}.$$

On every fixed compact set, $M|\operatorname{Im} \arcsin(z/M)| \rightarrow |\operatorname{Im} z|$ uniformly. This proves local boundedness and the first inequality in (23). Montel's theorem supplies convergent subsequences. Derivative convergence and (13) give

$$F'_N(0) = 1, \quad F'''_N(0) = -6S_N/M^2 \rightarrow -3/U^2.$$

List the positive critical points of P_N as $c_{N,j}$, $0 \leq j < N$, with $c_{N,0} = a_N$, and put $\beta_{N,j} = Mc_{N,j}$. Lemma 2.2 gives

$$F'_N(z) = \prod_{j=0}^{N-1} (1 - z^2/\beta_{N,j}^2), \quad \sigma_N := \sum_j \beta_{N,j}^{-2} = 3S_N/M^2 \rightarrow \frac{3}{2U^2} =: \sigma > 0. \quad (24)$$

Since α_N is the smallest $\beta_{N,j}$, $\alpha_N^2 \geq 1/\sigma_N$. If some subsequence had $\alpha_N \rightarrow \infty$, the logarithmic expansion of (24) on each fixed disk would give

$$\log F'_N(z) = -\sigma_N z^2 + o(1), \quad \sum_j \beta_{N,j}^{-4} \leq \sigma_N/\alpha_N^2 \rightarrow 0.$$

It would follow that $F_N \rightarrow \int_0^z e^{-\sigma t^2} dt$ locally uniformly. On the imaginary axis this limit has modulus $\int_0^y e^{\sigma t^2} dt$, contradicting the bound Ue^y . Thus (α_N) is bounded above as well. $\square$

Proposition 4.3. *Joint subsequential limits $F_N \rightarrow F$ locally uniformly and $\alpha_N \rightarrow a$ exist. Every such limit has $a > 0$ and satisfies*

$$D(z)F'(z)^2 = (z^2 - a^2)^2(U^2 - F(z)^2), \quad (25)$$

$$D(z) = z^4 + 2a^2(a^2 - U^2)z^2 + U^2a^4, \quad a^2 \geq 2U^2/3. \quad (26)$$

The quartic is positive on $\mathbb{R}$, and

$$F(x) = U \sin \phi(x), \quad \phi(x) = \int_0^x \frac{a^2 - t^2}{\sqrt{D(t)}} dt. \quad (27)$$

Here the real square root is positive. In particular, $|F(x)| = U$ at infinitely many real points.

Proof. Existence follows from Lemma 4.2. Fix any such limit. Equation (24) gives $a^2 \geq 2U^2/3$. The finite coefficients satisfy

$$M^4 B_N^f = U_N^2 \alpha_N^4, \quad M^2 A_N^f = 2\alpha_N^4 - 2U_N^2 \alpha_N^2 - 2U_N^2 \alpha_N^4/M^2.$$

Rescaling (20) in its polynomial form gives

$$(1 - z^2/M^2)(z^4 + M^2 A_N^f z^2 + M^4 B_N^f) F'_N(z)^2 = (z^2 - \alpha_N^2)^2 (U_N^2 - F_N(z)^2).$$

Taking the locally uniform limit proves (25).

Write $s = U^2$ and $b = a^2$. If $b \geq s$, the coefficients of D are nonnegative and its constant term is positive. If $b < s$, its minimum as a quadratic in $x^2 \geq 0$ is

$$b^2\{s - (s - b)^2\} \geq b^2s(1 - s/9) > 0,$$

because $b \geq 2s/3$ and $s \leq 3$. Thus (27) defines a real analytic function on the whole real axis. Near zero, (25) and $F'(0) = 1$ give the stated sine representation by differentiating $\arcsin(F/U)$. Real analytic continuation extends it across every critical point and contact. The integrand tends to -1 at positive infinity; after $x = a$, the phase decreases strictly to minus infinity. It crosses infinitely many odd multiples of $\pi/2$, proving the contact assertion. $\square$

5 The elliptic equation for the constant

Lemma 5.1. *The quartic D has four simple roots off the real and imaginary axes, and $|a^2 - U^2| < U$. The convergent integrals*

$$\theta = \int_0^\infty \left[1 + \frac{a^2 - t^2}{\sqrt{D(t)}} \right] dt, \quad (28)$$

$$\eta = \int_0^\infty \left[\frac{a^2 + t^2}{\sqrt{D(it)}} - 1 \right] dt \quad (29)$$

satisfy

$$\eta = 0, \quad \theta \in (2\mathbb{Z} + 1)\pi. \quad (30)$$

Both square roots in (28)–(29) are positive.

Proof. If $D(iy) = 0$, equation (25) forces $F(iy)^2 = U^2$, since $-y^2 - a^2 \neq 0$. Oddness and reality make $F(iy)$ purely imaginary, a contradiction. Hence $D(iy) > 0$ for all real y . Positivity on both axes implies that the quadratic in z^2 has nonreal conjugate roots. Its discriminant is negative, which is exactly $|a^2 - U^2| < U$, and its four roots are simple. The integrands in (28)–(29) are $O(t^{-2})$ at infinity. In particular

$$\phi(x) = -x + \theta + O(x^{-1}) \quad (x \rightarrow +\infty).$$

Write $F(iy) = if(y)$ for real y . The differential equation gives

$$f'(y)^2 = \frac{(a^2 + y^2)^2}{D(iy)} (U^2 + f(y)^2).$$

The derivative never vanishes and $f'(0) = 1$, so it is positive. Integrating with $f(0) = 0$ yields

$$F(iy) = iU \sinh \psi(y), \quad \psi(y) = \int_0^y \frac{a^2 + t^2}{\sqrt{D(it)}} dt = y + \eta + O(y^{-1}). \quad (31)$$

Outside a disk containing the roots of D , take the analytic square root with $\sqrt{D(z)} = z^2 + O(1)$ at infinity. The differential $(a^2 - z^2) dz / \sqrt{D(z)}$ has an even Laurent expansion $(-1 + O(z^{-2})) dz$. Its residue is zero, so it has a single-valued primitive in that exterior domain. Fix the integration constant by the real phase. The primitive then has the uniform expansion $\Phi_\infty(z) = -z + \theta +$

$O(z^{-1})$. The identity theorem extends $F = U \sin \Phi_\infty$ from the positive real ray to the exterior domain. Therefore

$$\lim_{y \rightarrow \infty} e^{-y} F(iy) = \frac{U}{2} (\sin \theta - i \cos \theta).$$

Equation (31) gives the same limit as $iUe^\eta/2$. Comparing parts gives $\sin \theta = 0$ and $-\cos \theta = e^\eta > 0$, proving (30). $\square$

Lemma 5.2. For $0 < k < 1$,

$$\int_0^\infty \frac{dx}{\sqrt{x^4 + 2(1 - 2k^2)x^2 + 1}} = K(k), \quad (32)$$

$$\int_0^\infty \left[1 - \frac{x^2}{\sqrt{x^4 + 2(1 - 2k^2)x^2 + 1}} \right] dx = 2E(k) - K(k). \quad (33)$$

For complementary moduli k and $q = \sqrt{1 - k^2}$, Legendre's relation [3, equation 19.7.1] is

$$K(k)E(q) + E(k)K(q) - K(k)K(q) = \pi/2. \quad (34)$$

Proof. For (32), substitute $x = \tan t$ to obtain the integral of $(1 - k^2 \sin^2(2t))^{-1/2}$ over $0 \leq t \leq \pi/2$. Symmetry gives $K(k)$.

Let $b = 1 - 2k^2$, $Q_b(x) = x^4 + 2bx^2 + 1$, and $I(b) = \int_0^\infty Q_b(x)^{-1/2} dx = K(k)$. Integration by parts in the left side $J(b)$ of (33) gives

$$J(b) = 2 \int_0^\infty \frac{x^2 + bx^4}{Q_b(x)^{3/2}} dx.$$

The substitution $x \mapsto 1/x$ identifies the integrals with numerators 1 and x^4 . Together with $I'(b) = -\int_0^\infty x^2 Q_b(x)^{-3/2} dx$, this gives

$$J(b) = bI(b) + 2(b^2 - 1)I'(b).$$

Differentiation of (2), with integration by parts, gives

$$\frac{dE}{dk} = \frac{E - K}{k}, \quad \frac{dK}{dk} = \frac{E/(1 - k^2) - K}{k}. \quad (35)$$

Since $dk/db = -1/(4k)$, the displayed expression for J becomes $2E(k) - K(k)$.

Differentiating the left side of (34) using (35) shows it is constant. As $k \rightarrow 0$, $K(k), E(k) \rightarrow \pi/2$, $E(q) \rightarrow 1$, and $(E(k) - K(k))K(q) \rightarrow 0$: the first factor is $O(k^2)$ and the second is $O(1 + |\log k|)$, directly from the integral definitions. The constant is therefore $\pi/2$. $\square$

Proof of the constant formula in Theorem 1.1. For the parameters in Proposition 4.3, set

$$\ell = a\sqrt{U}, \quad q^2 = \frac{U - U^2 + a^2}{2U}, \quad k = \sqrt{1 - q^2}. \quad (36)$$

Lemma 5.1 gives $q, k \in (0, 1)$. Scaling $t = \ell x$ in (28)–(29) and using Lemma 5.2 gives

$$\theta = \ell[2E(k) - (1 - 1/U)K(k)], \quad \eta = \ell[(1 + 1/U)K(q) - 2E(q)]. \quad (37)$$

Thus $\eta = 0$ implies $v(q) = K(q)/U > 0$. In particular $U > 1$, because $E(q) < K(q)$. Legendre's relation reduces the other offset to

$$\theta = \frac{\pi\ell}{K(q)}. \quad (38)$$

It is positive. Also

$$\ell < U\sqrt{U+1} < 2\sqrt{3} < 3\pi/2 \leq 3K(q),$$

where $|a^2 - U^2| < U$ and $U < 2$ were used. The odd-integer condition in (30) therefore forces $\theta = \pi$ and $\ell = K(q)$. Equations (36) now give

$$U = K(q)/v(q), \quad a^2 = K(q)v(q) = Uv(q)^2 = U(U - 1 + 2q^2),$$

which is (3).

It remains to verify uniqueness. The function K is strictly increasing, and (35) gives

$$v'(q) = \frac{(1 - 2q^2)E(q) - (1 - q^2)K(q)}{q(1 - q^2)} < 0.$$

For $q^2 \leq 1/2$, use $E(q) < K(q)$; for $q^2 > 1/2$, both terms in the numerator are negative. Since $v(0) = \pi/2$ and $v(q) \rightarrow -\infty$ as $q \rightarrow 1$, it has a unique zero q_0 . On $(0, q_0)$, each of K/v , $2q^2$, and $-v^2$ is strictly increasing. Hence

$$H(q) = K(q)/v(q) - 1 + 2q^2 - v(q)^2$$

is strictly increasing. Moreover $H(0) = -\pi^2/4$ and $H(q) \rightarrow +\infty$ as $q \rightarrow q_0$. This proves the unique root assertions. Every subsequential limit constructed above must realize this root, so the already existing constant in Proposition 3.2 is exactly (4). $\square$

6 Convergence and the gradient-descent lower bound

Theorem 6.1. *Let P_N be the optimizer in Proposition 2.3, and set*

$$a_*^2 = K(q_*)v(q_*), \quad U = K(q_*)/v(q_*).$$

The whole sequence satisfies

$$(2N + 1)a_N \longrightarrow a_*, \quad (2N + 1)P_N(z/(2N + 1)) \longrightarrow F_*(z) \quad (39)$$

locally uniformly on $\mathbb{C}$. The limit is entire, obeys $|F_(x + iy)| \leq Ue^{|y|}$, and has real values*

$$F_*(x) = U \sin \left(\int_0^x \frac{a_*^2 - t^2}{\sqrt{t^4 + 2a_*^2(a_*^2 - U^2)t^2 + U^2a_*^4}} dt \right). \quad (40)$$

Proof. From every subsequence, Lemma 4.2 gives a further joint convergent subsequence of the scaled critical points and polynomials. The proof of Theorem 1.1 fixes its parameters to U, a_* . The real phase then fixes its real values to (40); the identity theorem fixes the entire function. Thus all convergent subsequences have the same limit, and every subsequence has a further convergent subsequence. This proves (39) for the whole sequence. $\square$

Lemma 6.2 (Removing negative steps). *For every real schedule h , let $(h_+)_j = \max(h_j, 0)$. Then $W_{N,1}(h) \geq W_{N,1}(h_+)$.*

Proof. It suffices to replace one negative step $-b$, $b > 0$, by zero. Fix a differentiable convex scalar objective f and a minimizer x_* . Write $T_s(x) = x - sf'(x)$, and let P and S denote the compositions of the steps preceding and following this occurrence. They are continuous and fix x_* . For any allowed initial point x_0 , put $z = P(x_0)$ and $Q = T_{-b} \circ P$. Convexity implies $(z - x_*)f'(z) \geq 0$, so $Q(x_0) = z + bf'(z)$ lies at or beyond z on the ray from x_* . Since $Q(x_*) = x_*$, the intermediate value theorem on the segment joining x_* and x_0 supplies u in that segment with $Q(u) = z$. Thus $|u - x_*| \leq |x_0 - x_*|$ and

$$S(T_{-b}(P(u))) = S(z) = S(P(x_0)).$$

Every terminal point of the schedule with this step replaced by zero is attained by the original schedule, for the same function and an allowed initial point. Taking suprema proves the one-step inequality. Iteration over the negative entries proves the lemma. $\square$

Proof of Corollary 1.2. A scalar function extends to $\mathbb{R}^d$ by depending only on the first coordinate, so $W_{N,d}(h) \geq W_{N,1}(h)$. By Lemma 6.2 and the two tests in (1),

$$W_{N,d}(h) \geq W_{N,1}(h) \geq W_{N,1}(h_+) \geq \mathcal{B}_N(h_+) \geq E_N.$$

Taking the infimum over either schedule domain proves the first inequality in (8). Its last inequality uses (6) and the fact that $C/[(N+1/2)^2 + 4C/3]$ is strictly increasing in $C > 0$. $\square$

7 Rational certification

For $t = q^2$, the complete elliptic integrals have the convergent series

$$K(q) = \frac{\pi}{2} \sum_{j \geq 0} a_j t^j, \quad E(q) = \frac{\pi}{2} \left(1 - \sum_{j \geq 1} \frac{a_j t^j}{2j-1} \right), \quad a_0 = 1, \quad a_j = a_{j-1} \left(\frac{2j-1}{2j} \right)^2. \quad (41)$$

They follow by integrating the binomial series in (2). Since $0 < a_j \leq 1$, truncating after degree n leaves series tails bounded respectively by

$$\frac{t^{n+1}}{1-t}, \quad \frac{t^{n+1}}{(2n+1)(1-t)}. \quad (42)$$

The accompanying program `supplement/quadratic-huber/verify.py` uses $n = 60$ and rational arithmetic. It encloses π by alternating partial sums in

$$\pi = 16 \arctan(1/5) - 4 \arctan(1/239).$$

At the rational endpoints

$$q_- = 0.58413072912712, \quad q_+ = 0.58413072912713,$$

equations (41)–(42) enclose K, E, v with a strictly positive lower bound on v . The resulting rational intervals certify $H(q_-) < 0 < H(q_+)$. The monotonicity proved above encloses q_* between them. Since K/v is increasing while $v > 0$, the same intervals give (6). The program compares both displayed decimal bounds for C directly as rational numbers.

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