

An Explicit Support Bound for Counterexamples to Robin's Inequality

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Abstract

We prove unconditionally that every counterexample to Robin's inequality has more than $3.34 \cdot 10^{33}$ distinct prime factors. A support reduction and elementary optimality comparisons bound the size of a maximizer with $\omega(n)$ fixed. A primewise decomposition gives a positive term from the integer exponents, a nonlinear remainder, and an integrated sum over zeta zeros. Platt and Trudgian's verification controls the low zeros, and zero-density estimates bound the remaining tail. Arb interval arithmetic certifies positivity of the resulting lower bound throughout the support range. The quantitative inputs include recent preprints of Chirre–Helfgott and Bellotti–Trudgian–Yang.

1 Introduction

Let $\sigma(n)$ be the sum of the positive divisors of n , and let γ be Euler's constant. Robin [17] proved that the Riemann hypothesis is equivalent to

$$\sigma(n) < e^\gamma n \log \log n \quad (n \geq 5041).$$

Write $\omega(n)$ for the number of distinct prime factors of n . We give an explicit lower bound for $\omega(n)$ in any counterexample.

Theorem 1.1 (Counterexample support bound). *If $n \geq 5041$ and $\sigma(n) \geq e^\gamma n \log \log n$, then*

$$\omega(n) \geq 3341030598875664701253641258696900 > 3.34 \cdot 10^{33}. \quad (1.1)$$

The proof maximizes the normalized divisor sum with $\omega(n)$ fixed and reduces its support to an initial prime segment. Elementary optimality comparisons bound the size of this extremal integer. A primewise decomposition

then gives a positive term from the exponents, a nonlinear remainder, and an integrated sum over zeta zeros. Finite verification bounds the low zeros, while zero-density estimates and a zero-free region control the remaining tail.

Previous bounds. Vojak [19, Theorem 1.6(2)] gives $\omega(c) \geq 969672728$ for the least counterexample c . This part of his proof uses only the previously verified range of Robin’s inequality and an Euler-product bound, so it applies to every counterexample. Morrill and Platt [11, Theorem 5] verify Robin’s inequality through $10^{13.11485}$. This supplies the initial range for our support bound, after an Euler-product estimate converts a bound on $\omega(n)$ to a bound on the divisor ratio. We use Polak’s prime-power decomposition and positivity argument [16]. We give the proofs below and bound the integrated zero sum by zero-density estimates.

Quantitative inputs. Two inputs are recent preprints: the prime estimate of Chirre and Helfgott [4] and the zero-free region of Bellotti, Trudgian, and Yang [2]. The prime estimate incorporates Platt and Trudgian’s finite verification of RH [15]. Appendix A states the estimates and interval computation explicitly. Replacing the latter preprint by the published region of Mossinghoff, Trudgian, and Yang [12] gives $\omega(n) > 3.31 \cdot 10^{33}$, still using Chirre–Helfgott.

Outline. Section 2 gives the support reduction and size estimates. Section 3 bounds the prime contribution, bounds the integrated zero sum, and proves Theorem 1.1. Appendix A gives the quantitative estimates and completed interval calculation.

2 Extremal reduction

We normalize Robin’s inequality by

$$G(n) = \frac{\sigma(n)}{e^{\gamma} n \log \log n}.$$

Thus Robin’s inequality is $G(n) < 1$ for $n \geq 5041$ [17]. For $m \geq 1$, define

$$g_Z(m) = \sup_{\substack{n \geq 5041 \\ \omega(n)=m}} \log G(n).$$

If $g_{\mathbf{Z}}(m) < 0$, Robin's inequality holds for every $n \geq 5041$ with $\omega(n) = m$. For $m \geq 6$, the supremum is attained, as proved below, so the converse also holds.

We write p_i for the i th prime, and use

$$\pi(x) = \#\{p \leq x\}, \quad \theta(x) = \sum_{p \leq x} \log p, \quad \psi(x) = \sum_{p^j \leq x} \log p.$$

We use $\mathbf{Z}_{++} = \{1, 2, \dots\}$. The first reduction removes the choice of prime support [5, Section 5, Lemma 9].

Proposition 2.1 (Support reduction). *For every $m \geq 6$, the supremum defining $g_{\mathbf{Z}}(m)$ is a maximum. Every maximizer has the form*

$$n = \prod_{i=1}^m p_i^{r_i}, \quad r \in \mathbf{Z}_{++}^m.$$

Proof. Suppose

$$n = \prod_{i=1}^m q_i^{r_i}, \quad q_1 < \dots < q_m, \quad r_i \geq 1.$$

For a fixed exponent a , the local factor

$$1 + q^{-1} + \dots + q^{-a}$$

is strictly decreasing in q . We replace q_i by p_i for every i . This cannot decrease $\sigma(n)/n$ or increase n . Since $\log \log n$ is increasing on the range in question, the replacement cannot decrease $G(n)$.

For $m \geq 6$, the replacement remains feasible because

$$\prod_{i=1}^m p_i \geq \prod_{i=1}^6 p_i = 30030 > 5041.$$

If some $q_i > p_i$, the corresponding local factor increases strictly, while the denominator does not increase. Hence $G(n)$ increases strictly, and every maximizer uses $p_1, \dots, p_m$.

We now prove attainment. On this fixed support, $\sigma(n)/n$ is bounded above by

$$\prod_{i=1}^m \frac{p_i}{p_i - 1},$$

whereas $\log \log n \rightarrow \infty$ when any exponent tends to infinity. Hence $\log G(n) \rightarrow -\infty$ whenever an exponent tends to infinity. Only finitely many exponent vectors can lie above the value at $r_i = 1$, so the supremum is attained. $\square$

Lemma 2.2 (Size of an extremal integer). *Let $m \geq 6$, let $N_m^* = \prod_{p \leq p_m} p^{a_p}$ be a maximizer, and put $S = \log N_m^*$. Then*

$$S \leq \theta(p_m)^{\zeta(2)}. \quad (2.1)$$

For every p with $a_p \geq 2$,

$$p^{a_p} < 1 + S \log S. \quad (2.2)$$

Proof. The total possible increase in the log divisor ratio above the primorial is at most

$$- \sum_{p \leq p_m} \log(1 - p^{-2}) \leq \log \zeta(2).$$

Comparison with the primorial therefore gives

$$\log \log S \leq \log \log \theta(p_m) + \log \zeta(2),$$

which proves (2.1).

If $a_p \geq 2$, then N_m^*/p has the same support and remains above 5041. Optimality gives

$$\log \left(1 + \frac{1}{p + p^2 + \cdots + p^{a_p}} \right) \geq \log \log S - \log \log(S - \log p) > \frac{\log p}{S \log S}.$$

The last inequality follows by integrating the decreasing function $1/(u \log u)$ over $[S - \log p, S]$. On the other hand,

$$\frac{1}{\log p} \log \left(1 + \frac{1}{p + p^2 + \cdots + p^{a_p}} \right) = \int_{a_p-1}^{a_p} \frac{du}{p^{u+1} - 1} \leq \frac{1}{p^{a_p} - 1}.$$

Combining the inequalities proves (2.2). $\square$

3 An explicit lower bound

Throughout this section, put

$$X_0 = 10^9, \quad X_1 = 10^{20}, \quad X = 2.6918 \cdot 10^{35}, \quad T = 3 \cdot 10^{12}.$$

3.1 The primewise decomposition

Let $X_0 \leq x \leq X$ and $N = \prod_{p \leq x} p^{a_p}$, with $a_p \geq 1$, and put

$$S = \log N, \quad L = \log x, \quad z = \frac{S - x}{x}.$$

Define

$$R_\infty(x) = \int_x^\infty (\psi(t) - t) \frac{\log t + 1}{t^2 \log^2 t} dt,$$

$$B_2(x, S) = \log \left(1 + \frac{\log(1+z)}{L} \right) - \frac{z}{L}.$$

The classical error term in the prime number theorem ensures absolute convergence of the integral. For each $p \leq x$, put

$$k_p = \left\lfloor \frac{L}{\log p} \right\rfloor, \quad c_p = \frac{\log p}{xL}, \quad Q_{p,k} = \sum_{j > k} \frac{1}{jp^j},$$

and set

$$D(x, N) = \sum_{p \leq x} ((a_p - k_p)c_p - \log(1 - p^{-a_p-1}) - Q_{p,k_p}).$$

The following identity and positivity argument are due to Polak [16, Proposition 6 and Lemma 7].

Lemma 3.1 (Primewise decomposition). *For the above integer N ,*

$$-\log G(N) = R_\infty(x) + B_2(x, S) + D(x, N). \quad (3.1)$$

Every prime contribution to $D(x, N)$ is strictly positive.

Proof. Partial summation and Mertens' theorem give

$$\sum_{p^j \leq x} \frac{1}{jp^j} = \log L + \gamma + \frac{\psi(x) - x}{xL} - R_\infty(x).$$

The full Euler product adds $\sum_{p \leq x} Q_{p,k_p}$. Also

$$\psi(x) - \theta(x) = \sum_{p \leq x} (k_p - 1) \log p, \quad S - \theta(x) = \sum_{p \leq x} (a_p - 1) \log p.$$

Substitution gives (3.1). We use right-continuous prime-power sums; the value at a jump does not affect R_∞ .

For positivity, fix p , abbreviate $k = k_p$, $c = c_p$, and put

$$d_a = -\log(1 - p^{-a-1}), \quad u = \frac{1}{(k+1)p^{k+1}}.$$

Then $c > u$ and $Q_{p,k} \leq u/(1-1/p)$. If $a < k$, then $k \geq 2$, and the decreasing sequence $d_a - d_{a+1}$ satisfies

$$d_{k-1} - d_k > \frac{1}{1+p+\dots+p^k} \geq \frac{1}{kp^k} \geq c.$$

Consequently $(a-k)c + d_a > d_k$. At $a = k$, we have $d_k > (k+1)u \geq Q_{p,k}$. At $a = k+1$,

$$c + d_{k+1} > u \left(1 + \frac{k+1}{p}\right) \geq \frac{u}{1-1/p} \geq Q_{p,k}.$$

For $a \geq k+2$, the linear term alone exceeds $2u \geq Q_{p,k}$. These cases cover every integer $a \geq 1$. $\square$

Positivity allows us to discard the primes up to $v = \sqrt{x}$. For $p > v$, we have $k_p = 1$, and

$$(a_p - 1)c_p - \log(1 - p^{-a_p-1}) \geq \min\{p^{-2}, c_p\},$$

$$Q_{p,1} \leq \frac{1}{2p^2} + \frac{1}{3p^3(1-1/p)}.$$

The two alternatives switch at the unique $y > v$ satisfying $y^2 \log y = xL$. Retaining the primes on both sides of y yields an explicit bound

$$\sqrt{x}L D(x, N) \geq D_0(x), \tag{3.2}$$

whose formula is given in (A.4). The two integrals in Appendix A.2 retain both sides of the transition that gives $2\sqrt{2} - 2$. This constant also occurs in Ramanujan's asymptotic bound; see Nicolas [14, equation (1.2) and Theorem 1.1].

For a maximizer with $\omega(N) = m$, Lemma 2.2 supplies the size bound needed to estimate B_2 . The calculation in Appendix A, using this bound and explicit prime estimates, proves, throughout $[X_0, X]$,

$$0 \leq \log N_m^* - \theta(x) < 40\sqrt{x}, \quad x = p_m, \tag{3.3}$$

and gives

$$\sqrt{x}L B_2(x, \log N_m^*) \geq -C_B(x). \tag{3.4}$$

3.2 The zeros above the verified height

Let $\rho = \beta + i\tau$ run over the nontrivial zeros, counted with multiplicity. Set

$$h(u) = \frac{u+1}{u^2}, \quad J_\rho(L) = \int_L^\infty e^{-(1-\rho)u} h(u) du.$$

The integrated von Mangoldt formula used by Polak [16, Section 5.3] gives

$$\begin{aligned} R_\infty(x) = & - \sum_\rho \frac{J_\rho(L)}{\rho} - \frac{\log(2\pi)}{xL} \\ & - \frac{1}{2} \int_x^\infty \log(1-t^{-2}) \frac{\log t + 1}{t^2 \log^2 t} dt. \end{aligned} \tag{3.5}$$

The last term is positive. The zero sum is absolutely convergent. On a finite interval, integration by parts bounds the integrated zero term by $O_L(|\tau|^{-2})$, uniformly in the upper endpoint. The same bound holds for smooth approximations with uniformly bounded variation. Applying the explicit formula to these approximations and then taking limits justifies (3.5).

Lemma 3.2 (Integrated zero kernel). *For every nontrivial zero ρ ,*

$$\left| \frac{J_\rho(L)}{\rho} \right| \leq \frac{x^{\beta-1} h(L)}{|\rho| |1-\rho|} \leq \frac{x^{\beta-1} h(L)}{\tau^2}. \tag{3.6}$$

Proof. The positive Laplace representation

$$h(u) = \int_0^\infty (1+t) e^{-ut} dt$$

and Fubini's theorem give

$$J_\rho(L) = x^{\rho-1} \int_0^\infty \frac{(1+t) e^{-Lt}}{1+t-\rho} dt.$$

Since $|1+t-\rho| \geq |1-\rho|$ for $t \geq 0$, taking absolute values proves the result. $\square$

Platt and Trudgian's verification gives $\beta = 1/2$ for $|\tau| \leq T$. The classical identity [13, proof of Theorem 3(b)]

$$Z := \sum_\rho \frac{1}{\rho(1-\rho)} = 2 + \gamma - \log(4\pi)$$

bounds the reciprocal sum over those zeros from above. Indeed, every omitted conjugate pair has positive real contribution, since

$$\Re \frac{1}{\rho(1-\rho)} = \frac{\tau^2 + \beta(1-\beta)}{|\rho(1-\rho)|^2} > 0.$$

After multiplication by $\sqrt{x}L$, the absolute contribution of the low zeros and the constant term is at most

$$C_{\text{low}}(x) = Z \left(1 + \frac{1}{L} \right) + \frac{\log(2\pi)}{\sqrt{x}}.$$

For positive ordinates above T , define

$$S_2 = \sum_{\tau > T} \tau^{-2}, \quad A_2(s) = \sum_{\substack{\tau > T \\ \beta > s}} \tau^{-2}.$$

Reflection symmetry yields the exact identity

$$\sum_{|\tau| > T} \frac{x^{\beta-1/2}}{\tau^2} = 2S_2 + 2 \int_{1/2}^1 L(x^{s-1/2} - x^{1/2-s}) A_2(s) ds. \quad (3.7)$$

Each pair $\beta, 1-\beta$ contributes its value at $1/2$ plus the integral of its derivative; zeros on the critical line contribute only to the first term.

Appendix A.3 bounds S_2 by S_2^+ and $A_2(s)$ by a constant a_i on each interval $[s_i, s_{i+1}]$. Define

$$\begin{aligned} W_+(x) &= 2S_2^+ + 2 \sum_i a_i (x^{s_{i+1}-1/2} + x^{1/2-s_{i+1}} \\ &\quad - x^{s_i-1/2} - x^{1/2-s_i}), \\ C_{\text{high}}(x) &= \left(1 + \frac{1}{L} \right) W_+(x). \end{aligned}$$

Equations (3.6) and (3.7) then give

$$\sqrt{x}L R_\infty(x) \geq -C_{\text{low}}(x) - C_{\text{high}}(x).$$

Combining the bounds, every maximizer with $X_0 \leq p_m \leq X$ satisfies

$$-\sqrt{x}L g_{\mathbf{Z}}(m) \geq M(x) := D_0(x) - C_B(x) - C_{\text{low}}(x) - C_{\text{high}}(x). \quad (3.8)$$

Proof of Theorem 1.1. The interval calculation in Appendix A.4 proves $M(x) > 0$ throughout $[X_0, X]$. By Proposition 2.1 and (3.8), Robin's inequality holds whenever $\omega(n) = m$ and $X_0 \leq p_m \leq X$.

For $p_m \leq X_0$, put $N_0 = 10^{10^{13.11485}}$. Morrill and Platt [11, Theorem 5] verify Robin's inequality for $5040 < n \leq N_0$. For $n > N_0$, each local factor is smaller than $q/(q-1)$, which decreases with q . Thus $\omega(n) = m$ bounds the divisor ratio by the Euler product over the first m primes. The bound of Rosser and Schoenfeld [18, Theorem 8] then gives

$$\frac{\sigma(n)}{n} < \prod_{p \leq X_0} (1 - 1/p)^{-1} < e^\gamma \left(\log X_0 + \frac{1}{2 \log X_0} \right) < e^\gamma \log \log N_0.$$

The last comparison is included in the certificate. This argument applies also when $m < 6$.

Every counterexample therefore has $\omega(n) \geq \pi(X) + 1$. The lower bound for $\pi(X)$ in (A.9), evaluated with outward rounding, gives (1.1). $\square$

4 Conclusion

The bound $M(x) > 0$ on $[X_0, X]$, together with the estimate for $p_m \leq X_0$, excludes every counterexample with at most $3.34 \cdot 10^{33}$ distinct prime factors.

At $x = X$, the largest subtraction in $M(x)$ is $C_{\text{high}}(x)$. Improved density estimates for the unverified zeros can be used in the same calculation.

A The finite computation

The verification supplement¹ contains the interval calculation for Theorem 1.1. It uses Arb [9] through `python-flint` 0.9.0. An unresolved comparison terminates the computation without certifying the bound. The source, density tables, parameters, and verified interval covers are in the `verify/omega/` directory of the source repository.

A.1 Prime estimates and size control

Retain the constants X_0, X_1, X, T of Section 3, and put

$$a_0 = \frac{\pi}{T}, \quad b_0 = 115.1, \quad C_\theta = 1.01624.$$

¹Version 0.2.0: <https://github.com/epicycloids/robin-integrality-gap/tree/v0.2.0/verify>

Rosser and Schoenfeld [18, Theorems 9 and 13] give $\theta(t) < C_\theta t$ and $0 \leq \psi(t) - \theta(t) < 1.4262\sqrt{t}$. Chirre and Helfgott [4, Corollary 1.3] give $|\psi(t) - t| \leq a_0 t + 113.67\sqrt{t}$ for $t \geq 1$, whence

$$|\theta(t) - t| \leq a_0 t + b_0 \sqrt{t}. \quad (\text{A.1})$$

Their finite-height hypothesis requires RH through $T + 1 + \pi/3$, which is below Platt and Trudgian's verified height 3000175332800.

For the smaller range we use Büthe's estimate under partial verification of RH [3], in the form stated for θ by Johnston [10, Proposition 3.1]:

$$|\theta(t) - t| \leq \frac{\sqrt{t} \log^2 t}{8\pi} \quad (599 \leq t \leq 2.169 \cdot 10^{25}). \quad (\text{A.2})$$

We only use it through X_1 . The certificate also checks $4.92\sqrt{X_1/\log X_1} < T$.

Let $S = \log N_m^*$, $x = p_m$, and $V = 1 + S \log S$. Lemma 2.2 gives $p^k < V$ whenever $2 \leq k \leq a_p$. Summing over $k \geq 2$ gives

$$S - \theta(x) \leq B(S) := C_\theta \left(\sqrt{V} + \frac{\log V}{\log 2} V^{1/3} \right).$$

The terms with $k = 2$ contribute at most $C_\theta \sqrt{V}$; for $k \geq 3$, the primes are at most $V^{1/3}$. On each interval $[u, v]$ of a doubling cover of $[X_0, X]$, start with

$$s_0 = (C_\theta v)^{\zeta(2)}, \quad s_{j+1} = C_\theta v + B(s_j).$$

The bound (2.1) and monotonicity of B give $S \leq s_j$ by induction. The certificate checks $B(s_{20}) < 40\sqrt{u}$ on every interval, proving (3.3).

Define

$$\delta(x) = \begin{cases} (L^2/(8\pi) + 40)/\sqrt{x}, & x \leq X_1, \\ a_0 + 155.1/\sqrt{x}, & x > X_1, \end{cases}$$

$$\varepsilon(x) = \begin{cases} (L/2)^2/(8\pi x^{1/4}), & x \leq X_1, \\ a_0 + b_0/x^{1/4}, & x > X_1. \end{cases}$$

Then $|(S - x)/x| \leq \delta(x)$, and $|\theta(t) - t| \leq \varepsilon(x)t$ uniformly for $\sqrt{x} \leq t \leq x$. For the latter assertion, note that $\log^2 t/\sqrt{t}$ decreases for $t > e^4$. Both pairs of estimates are valid at X_1 .

Put $A_- = L + \log(1 - \delta)$. The second derivative of $g(z) = \log(L + \log(1 + z)) - \log L$ satisfies

$$-g''(z) = \frac{L + \log(1 + z) + 1}{(1 + z)^2 (L + \log(1 + z))^2}.$$

This expression decreases where $L + \log(1 + z) > 0$. Taylor's theorem therefore proves (3.4) with

$$C_B(x) = \frac{\sqrt{x}L\delta^2}{2} \frac{A_- + 1}{(1 - \delta)^2 A_-^2}. \quad (\text{A.3})$$

The interval calculation checks $\delta < 1$ and $A_- > 0$.

A.2 The explicit prime contribution

Put $v = \sqrt{x}$, let $y^2 \log y = xL$, and write

$$w = \log y = \tfrac{1}{2}W(2xL), \quad r = y/v,$$

where W is the Lambert W -function. Define

$$f(t) = \min\{t^{-2}, \log t/(xL)\} - \frac{1}{2t^2}, \quad \Phi(t) = \frac{f(t)}{\log t}.$$

The function Φ increases from zero on $[v, y]$ and decreases on $[y, x]$, with $\Phi(y) = 1/(2xL)$. Integration by parts against $\theta(t) - t$ gives the error bound

$$-\varepsilon x \Phi(x) - \varepsilon \int_v^x t |\Phi'(t)| dt.$$

Consequently,

$$\sum_{v < p \leq x} f(p) \geq (1 + \varepsilon)I_s + (1 - \varepsilon)I_t - 2\varepsilon y \Phi(y),$$

where $I_s = \int_v^y \Phi(t) dt$ and $I_t = \int_y^x \Phi(t) dt$. With $E_1(u) = \int_u^\infty e^{-t} dt/t$, their scaled values are

$$J_s := vLI_s = r - 1 - \frac{vL}{2}(E_1(L/2) - E_1(w)),$$

$$J_t := vLI_t = \frac{vL}{2}(E_1(w) - E_1(L)).$$

The omitted cubic terms are at most

$$\frac{1}{3(1 - 1/v)} \sum_{n > v} n^{-3} \leq \frac{1}{6(1 - 1/v)(v - 1)^2}.$$

Lemma 3.1 now proves (3.2) with

$$D_0(x) = (1 + \varepsilon)J_s + (1 - \varepsilon)J_t - \varepsilon r - \frac{L}{6v(1 - 1/v)^3}. \quad (\text{A.4})$$

A.3 Density bounds for zeros of large height

Let $N(t)$ count zeros with $0 < \tau \leq t$, and let $N(s, t)$ count those with $\beta > s$ as well. Hasanalizade, Shen, and Wong [7, Corollary 1.2] prove

$$\left| N(t) - \frac{t}{2\pi} \log \frac{t}{2\pi e} \right| \leq A \log t + B \log \log t + C \quad (t \geq e),$$

where $(A, B, C) = (0.1038, 0.2573, 9.3675)$. Partial summation gives

$$S_2 \leq S_2^+ := \frac{\log(T/(2\pi)) + 1}{2\pi T} + \frac{2A \log T + A/2 + 2B \log \log T + B/(2 \log T) + 2C}{T^2}. \quad (\text{A.5})$$

In the error integral, use $\log(\log T + u) \leq \log \log T + u/\log T$. Reflection symmetry gives $A_2(s) \leq S_2^+/2$ for $s \geq 1/2$.

We use the zero-free region

$$\beta \leq 1 - \frac{1}{R \log |\tau|}, \quad R = 4.896,$$

from [2, Theorem 1]. It applies for $|\tau| \geq 3$. Thus a zero counted by $A_2(s)$ has

$$\tau > U(s) := \max\{T, \exp(1/(R(1-s)))\}.$$

Dropping the nonpositive boundary term in partial summation gives

$$A_2(s) \leq 2 \int_{U(s)}^{\infty} \frac{N(s, t)}{t^3} dt. \quad (\text{A.6})$$

The interval constants in Fiori, Kadiri, and Swidinsky [6, Corollary 2.9 and Table 7] give

$$N(s, t) \leq c_1 t^{(8/3)(1-s)} (\log t)^{5-2s} + c_2 (\log t)^2 \quad (t \geq T).$$

The supplement transcribes both parts of their table, using each pair (c_1, c_2) only on its stated s -interval. The second part uses the subconvexity constant 0.618, established by Hiary, Patel, and Yang [8, Theorem 1.1]. We do not interpolate pointwise density constants.

For an applicable row, put $p = (8/3)(1-s)$, $q = 5 - 2s$, and $\ell = \log U(s)$. Its contribution to (A.6) is exactly

$$\frac{2c_1 \Gamma(q+1, (2-p)\ell)}{(2-p)^{q+1}} + \frac{c_2}{U(s)^2} \left(\ell^2 + \ell + \frac{1}{2} \right), \quad (\text{A.7})$$

where $\Gamma(a, z)$ is the upper incomplete gamma function. Arb encloses the entire integral, including its infinite tail.

For $s \geq 0.9927$, we also integrate the first 18 rows of Bellotti's log-free estimate [1, Theorem 1.1 and Table 2]. These rows cover $T < t \leq e^{46.2}$, and each is valid on the stated smaller s -range. A row $N(s, t) \leq Ct^{B(1-s)}$, between logarithmic endpoints ℓ_0, ℓ_1 , contributes

$$\frac{2C}{d} \left(e^{-d\ell_0} - e^{-d\ell_1} \right), \quad d = 2 - B(1 - s) > 0.$$

The lower endpoint is raised to $\log U(s)$ when necessary, and an empty interval contributes zero. Formula (A.7) bounds the entire remaining tail beyond $e^{46.2}$.

At each grid point we take the least of these applicable upper bounds and $S_2^+/2$, then the least of all bounds obtained at earlier grid points. This remains an upper bound because $A_2(s)$ decreases. The bound a_i at the left endpoint is valid throughout $[s_i, s_{i+1}]$. We use $[1/2, 3/5]$ as the first interval and steps of $1/200000$ thereafter up to 1. The weight integral in (3.7) is evaluated exactly on each interval.

A.4 Covering the support range

On a real interval $[u, v]$, the program evaluates $D_0 - C_B$ by Arb interval extension. The function C_{low} decreases, and W_+ increases. Therefore a lower enclosure of M is obtained from

$$(D_0 - C_B)([u, v]) - C_{\text{low}}(u) - \left(1 + \frac{1}{\log u} \right) W_+(v). \quad (\text{A.8})$$

Starting with decade intervals, the program splits at X_1 and then at integer geometric midpoints until every interval has a strictly positive lower enclosure. Exact endpoint comparisons check that the resulting intervals cover $[X_0, X]$ without gaps.

At 192-bit precision this gives 499 intervals, with every saved lower bound exceeding $1.14 \cdot 10^{-7}$. A second calculation at 256 bits and twice the density-grid resolution gives 468 intervals, with every lower bound exceeding $1.88 \cdot 10^{-6}$. Both certify the same endpoint X . At 192 bits, the

endpoint contributions are

term	value, rounded
$D_0(X)$	0.807339956359
$C_B(X)$	0.000000288129
$C_{\text{low}}(X)$	0.046757623230
$C_{\text{high}}(X)$	0.760579553051
$M(X)$	0.000002491949

The displayed values summarize the saved enclosures; the interval comparisons use the full enclosures.

Finally, partial summation of $\theta(t) \geq (1 - a_0)t - b_0\sqrt{t}$ for $t \geq 2$ gives

$$\begin{aligned} \pi(X) &\geq P_-(X) := q(X) - q(2) + \frac{2(1 - a_0) - b_0\sqrt{2}}{\log 2}, \\ q(t) &= (1 - a_0) \text{li}(t) - \frac{b_0}{2} \text{li}(\sqrt{t}). \end{aligned} \tag{A.9}$$

We take the ceiling of a downward-rounded enclosure of $P_-(X)$ and add one. This proves the integer in (1.1), without computing $\pi(X)$ exactly.

With $R = 5.558691$ from [12, Theorem 1.1], a separate 192-bit calculation covers $[X_0, 2.6690 \cdot 10^{35}]$ in 492 intervals and gives

$$\omega(n) \geq 3313081353157540211029778091609681.$$

All three calculations include the size bounds, the comparison using Morrill and Platt's result for $p_m \leq X_0$, and the sums over zeros of arbitrarily large height. Separate 75-digit evaluations check the special-function expressions against direct quadrature and test the reflection and kernel formulas. These evaluations are numerical checks; the proof uses the interval comparisons.

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