

Robin Extrema: Computation and the Cost of Integral Exponents

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Abstract

We study the largest normalized divisor sum in Robin’s inequality among integers with exactly m distinct prime factors. For $m \geq 6$, every maximizer uses the first m primes and is a prefix of the exponent increments ordered by marginal gain per unit log-size. Assuming exact comparisons, including equality, an explicit stopping bound gives a finite algorithm. We also treat bounded supports and certify the maxima for $m \leq 5$.

The optimizer for real exponents is determined by one scalar equation. For all sufficiently large m , each exponent of an integer maximizer equals the floor or ceiling of its relaxed value. If $g_{\mathbf{Z}}(m)$ and $g_{\mathbf{R}}(m)$ are the two optimal logarithmic values, then

$$g_{\mathbf{R}}(m) - g_{\mathbf{Z}}(m) = \frac{2\sqrt{2} + o(1)}{\sqrt{p_m} \log p_m}.$$

The two equal leading contributions come from downward and upward rounding near the transition between exponents one and two. The common coefficient in the Axler–Nicolas comparison of size-constrained maxima arises from the same second-exponent threshold equation. Under the Riemann hypothesis, the relaxation eventually violates Robin’s inequality while the integer optimum satisfies it. If the hypothesis fails, two-sided Mertens oscillations dominate the gap.

1 Introduction

Let $\sigma(n)$ be the sum of the positive divisors of n , and let γ be Euler’s constant. Grönwall [7] proved

$$\limsup_{n \rightarrow \infty} \frac{\sigma(n)}{e^{\gamma} n \log \log n} = 1.$$

Robin [18] sharpened this maximal-order statement to a pointwise criterion. The Riemann hypothesis holds if and only if

$$\sigma(n) < e^\gamma n \log \log n \quad (n \geq 5041).$$

We maximize the normalized ratio over integers with a fixed number of distinct prime factors. We characterize the integer optimizers and compare them with the corresponding real-exponent relaxation.

Write $\omega(n)$ for the number of distinct prime factors of n . For each m , let $g_{\mathbf{Z}}(m)$ be the largest value of the logarithm of this ratio among integers $n \geq 5041$ with $\omega(n) = m$. For $m \geq 6$, a support exchange reduces the problem to the first m primes.

Integer computation. Every integer optimizer is a prefix of the exponent increments ordered by Nicolas's score $F(p, k)$, with equal scores taken as one block. Every maximizing prefix satisfies

$$\log N \leq \theta(p_m)^{\zeta(2)}.$$

The characterization and stopping bound give a finite computation, assuming exact comparisons, including equality. The verification supplement uses interval arithmetic to certify the small-support maxima and the transition in the bounded-support problem $\omega(n) \leq m$.

Integrality gap. Allowing the exponents on the first m primes to be real numbers at least one gives a relaxation with value $g_{\mathbf{R}}(m)$. We prove unconditionally that

$$\Delta(m) := g_{\mathbf{R}}(m) - g_{\mathbf{Z}}(m) = \frac{2\sqrt{2} + o(1)}{\sqrt{p_m} \log p_m}. \quad (1.1)$$

The real optimizer is determined by one scalar equation. For all sufficiently large m , each exponent of an integer maximizer is one of the two integers adjacent to its relaxed counterpart. We approximate the sum of the local rounding losses by an integral using the prime number theorem. Only the transition between exponents one and two contributes at first order. Rounding down and rounding up each contribute $\sqrt{2}/(\sqrt{p_m} \log p_m)$.

Robin's inequality. The identity $-g_{\mathbf{Z}}(m) = \Delta(m) - g_{\mathbf{R}}(m)$ separates the rounding loss from the analytic Mertens term. Under the Riemann hypothesis, Nicolas's integrated explicit formula makes the relaxed value eventually positive and bounds its normalized coefficient above by $4 + \gamma - \log(4\pi)$.

This is smaller than the gap coefficient $2\sqrt{2}$, so the integer optimum is eventually below Robin’s bound. If the hypothesis fails, Nicolas’s two-sided Mertens oscillations dominate (1.1). It follows that the hypothesis is equivalent to Robin’s inequality holding on every sufficiently large fixed- ω slice.

1.1 Relation to previous work

Superabundant and colossally abundant numbers. Superabundant (SA) numbers are the image of the record map

$$X \mapsto \operatorname{argmax}_{n \leq X} \frac{\sigma(n)}{n},$$

and colossally abundant (CA) numbers are the image of the map

$$\varepsilon \mapsto \operatorname{argmax}_{n \geq 1} \frac{\sigma(n)}{n^{1+\varepsilon}},$$

defined for $\varepsilon > 0$. The theory of these numbers goes back to Ramanujan [15] and Alaoglu and Erdős [2]. Nicolas developed the benefit method [10, 11], which Robin applied to related integer optimization problems [16, 17]. Nicolas’s marginal score $F(p, k)$ governs whether the exponent of p reaches k [13, Section 4.1]. Briggs [4, Section 3.1] computes CA numbers by advancing through the exponent-change scores, with compact storage of the exponents. Our prefix characterization applies this ordering to the fixed-support Robin objective and supplies a finite stopping bound.

In the knapsack formulation, $F(p, k)$ is the gain in the log divisor ratio divided by the increase in log-size. Fixing $\omega(n) = m$ makes the first-exponent increments $F(p, 1)$ mandatory for $p \leq p_m$, while the optional increments with $k \geq 2$ retain their classical order. The resulting optimizer is CA and SA relative to its prescribed support, but need not be a global CA or SA number. In the bounded-support problem, the first-exponent increments are optional.

Classical Robin asymptotics. Under the Riemann hypothesis, Ramanujan’s expansion for generalized superior highly composite numbers, in the corrected version published by Nicolas and Robin, contains the normalized constant $2\sqrt{2} - 2$ and the zeta-zero term that appear in the classical Robin margin [15, formula (382) and Note 71]. Nicolas gives a modern effective treatment for CA numbers and then for all integers [13, Proposition 5.1 and Theorem 1.1]. In the fixed-support problem, the optimizer characterization

and rounding formula give the unconditional integrality gap. Nicolas’s integrated explicit formula then determines the analytic term in the Robin margin.

The Axler–Nicolas comparison. Axler and Nicolas [3] prove

$$\frac{\max_{n \leq X} n/\varphi(n)}{\max_{n \leq X} \sigma(n)/n} = 1 + \frac{2\sqrt{2}}{\sqrt{\log X} \log \log X} + O\left(\frac{1}{\sqrt{\log X} (\log \log X)^2}\right),$$

where φ is Euler’s totient function. Nicolas obtains a full expansion [14, Theorem 1.2]. The second-exponent threshold equation is the same in both problems. Section 6.1 compares the size constraint with fixed support and identifies the Euler-factor and size contributions.

Other reductions. The first-prime support exchange is standard in work on possible counterexamples to Robin’s inequality [5, Section 5, Lemma 9]. Other work restricts a least counterexample or reduces Robin’s criterion to record subsequences [1, 9].

Outline. Section 2 gives the support reduction and prefix characterization. Section 3 states the finite algorithm and its certification procedure. Section 4 treats small and bounded supports. Section 5 gives the scalar relaxed optimizer and one-step proximity. Section 6 evaluates the rounding loss and relates it to the classical CA exponent thresholds. Section 7 adds Nicolas’s analytic input and derives the fixed-slice criterion. Appendix A gives the finite computations.

2 The fixed- ω extremal problem

We normalize Robin’s inequality by

$$G(n) = \frac{\sigma(n)}{e^\gamma n \log \log n}.$$

Thus Robin’s inequality is $G(n) < 1$ for $n \geq 5041$ [18]. For $m \geq 1$, define the fixed- ω extremal value

$$g_Z(m) = \sup_{\substack{n \geq 5041 \\ \omega(n)=m}} \log G(n).$$

A negative value of $g_{\mathbf{Z}}(m)$ means that Robin's inequality holds throughout the slice. For $m \geq 6$, the attainment proved below makes the converse true as well.

We write p_i for the i th prime and use

$$\pi(x) = \#\{p \leq x\}, \quad \theta(x) = \sum_{p \leq x} \log p.$$

We use $\mathbf{Z}_{++} = \{1, 2, \dots\}$. The first reduction restricts the support to the first m primes [5, Section 5, Lemma 9].

Proposition 2.1 (Support reduction). *For every $m \geq 6$, the supremum defining $g_{\mathbf{Z}}(m)$ is a maximum. Every maximizer has the form*

$$n = \prod_{i=1}^m p_i^{r_i}, \quad r \in \mathbf{Z}_{++}^m.$$

Proof. Suppose

$$n = \prod_{i=1}^m q_i^{r_i}, \quad q_1 < \dots < q_m, \quad r_i \geq 1.$$

For a fixed exponent a , the local factor

$$1 + q^{-1} + \dots + q^{-a}$$

is strictly decreasing in q . We replace q_i by p_i for every i . This cannot decrease $\sigma(n)/n$ or increase n . Since $\log \log n$ is increasing on the range in question, the replacement cannot decrease $G(n)$.

For $m \geq 6$, the replacement remains feasible because

$$\prod_{i=1}^m p_i \geq \prod_{i=1}^6 p_i = 30030 > 5041.$$

If some $q_i > p_i$, the corresponding local factor increases strictly, while the denominator does not increase. Hence $G(n)$ increases strictly, and every maximizer uses $p_1, \dots, p_m$.

We now prove attainment. On this fixed support, $\sigma(n)/n$ is bounded above by

$$\prod_{i=1}^m \frac{p_i}{p_i - 1},$$

whereas $\log \log n \rightarrow \infty$ when any exponent tends to infinity. Hence $\log G(n) \rightarrow -\infty$ whenever an exponent tends to infinity. Only finitely many exponent vectors can lie above the value at $r_i = 1$, so the supremum is attained. $\square$

2.1 The integer optimizer

Fix $m \geq 6$, with support $p_1, \dots, p_m$.

For $x > 1$ and an integer $k \geq 1$, define the standard CA score [13, Section 4.1]

$$F(x, k) = \frac{1}{\log x} \log \left(1 + \frac{1}{x + x^2 + \dots + x^k} \right). \quad (2.1)$$

Raising the exponent of p from $k-1$ to k changes $\log(\sigma(n)/n)$ by

$$\log \left(\frac{\sigma(p^k)/p^k}{\sigma(p^{k-1})/p^{k-1}} \right) = \log p F(p, k). \quad (2.2)$$

Equivalently,

$$F(p, k) = \int_{k-1}^k \frac{du}{p^{u+1} - 1}. \quad (2.3)$$

This shows that the score decreases strictly in both arguments and that

$$F(p, k) \leq \frac{1}{p^k - 1}.$$

In particular, only finitely many eligible scores exceed any positive level. The fixed-support problem includes the increments (p, k) with $p \leq p_m$ and $k \geq 2$. Their log-size weights are $\log p$, and their scores are ordered as above.

Group equal scores and order the resulting blocks by decreasing score. Starting from the primorial $\prod_{p \leq p_m} p$, let N_j be the integer obtained after the first j blocks. We refer to these integers as score-ordered prefixes. Every such prefix is a valid exponent vector because each predecessor in a prime chain has a larger score.

Theorem 2.2 (Prefix characterization). *Every integer maximizer is one of the prefix integers N_j . If N_m^* is any maximizer and*

$$L_m = \log N_m^*, \quad \varepsilon_m = \frac{1}{L_m \log L_m},$$

then an increment (p, k) , $k \geq 2$, is selected exactly when

$$F(p, k) > \varepsilon_m. \quad (2.4)$$

No eligible score equals ε_m , and no maximizer cuts a tie block. Every maximizing prefix also satisfies

$$\log N_m^* \leq \theta(p_m)^{\zeta(2)}. \quad (2.5)$$

Proof. Consider the piecewise-linear curve through the points

$$\left(\log N_j, \log \frac{\sigma(N_j)}{N_j} \right).$$

Its slopes are the ordered scores. This curve is a fractional-knapsack upper bound. Let $\mathcal{E}$ be any finite selection of increments with total extra log-size z . On an interval with score c , let $\mathcal{P}$ be the set of all increments in earlier blocks. Then

$$\begin{aligned} \sum_{(p,k) \in \mathcal{E}} \log p F(p,k) &= cz + \sum_{(p,k) \in \mathcal{E}} \log p (F(p,k) - c) \\ &\leq cz + \sum_{(p,k) \in \mathcal{P}} \log p (F(p,k) - c). \end{aligned}$$

The right-hand side is the linear interpolation between the two adjacent score-ordered prefix values.

On each interpolation interval the log divisor ratio is affine in $\log n$, while $-\log \log \log n$ is strictly convex. One of the two prefix endpoints therefore has at least as large a value of $\log G$ as any integer selection in the interval. Thus the supremum over integer selections equals the supremum over score-ordered prefixes.

The total possible gain above the primorial is

$$- \sum_{p \leq p_m} \log(1 - p^{-2}) \leq \log \zeta(2).$$

It follows that the prefix objective tends to minus infinity, so a maximizing prefix exists. Comparing any maximizing prefix with the primorial gives

$$\log \log \log N_m^* \leq \log \log \theta(p_m) + \log \zeta(2),$$

which is (2.5).

It remains to characterize an arbitrary maximizer N_m^* . The function $u \mapsto \log \log u$ is strictly concave, and its derivative at L_m is ε_m . Optimality and the tangent inequality give, for every integer M on the prescribed support,

$$\begin{aligned} \log \frac{\sigma(M)}{M} - \log \log \log M &\leq \log \frac{\sigma(N_m^*)}{N_m^*} - \log \log L_m, \\ \log \log \log M &\leq \log \log L_m + \varepsilon_m (\log M - L_m). \end{aligned}$$

Consequently,

$$\log \frac{\sigma(M)}{M} - \varepsilon_m \log M \leq \log \frac{\sigma(N_m^*)}{N_m^*} - \varepsilon_m L_m.$$

This expression is separable over the optional increments. Equation (2.2) shows that an increment is selected when its score exceeds ε_m and is unselected when its score is smaller. Equality cannot occur. Adding or removing an increment with score ε_m would preserve the last display, while strictness in the tangent inequality would produce a larger value of $\log G$. Thus the selected increments form a score-ordered prefix and satisfy (2.4). This also proves the tie assertion and extends (2.5) to every maximizer. $\square$

The threshold gives the following SA and CA interpretation. The integer N_m^* maximizes

$$\frac{\sigma(n)}{n^{1+\varepsilon_m}}$$

over integers on the prescribed support. It consequently maximizes $\sigma(n)/n$ on the same support under the size constraint $\log n \leq L_m$. It need not be a global CA or SA number because the first-exponent increments $F(p, 1)$ are fixed by the support constraint.

3 Computing the extrema

The prefix characterization gives a finite algorithm in a model permitting exact comparisons, including equality, of scores, objective values, and stopping bounds. Its input is $m \geq 6$, and its output is the set of maximizing exponent vectors, together with $g\mathbf{z}(m)$.

Put

$$\theta_m = \theta(p_m), \quad B_m = \theta_m^{\zeta(2)}.$$

We maintain the current exponents a_i , log-size S , and log divisor ratio A . The available score in the i th chain is $F(p_i, a_i + 1)$.

1. Initialize

$$a_i = 1, \quad S = \theta_m, \quad A = \sum_{i=1}^m \log(1 + p_i^{-1}).$$

Set the best value to $A - \gamma - \log \log S$, and retain the primorial exponent vector.

2. Let c be the largest available score, and let I contain all indices with score c . For each $i \in I$, increase a_i by one and update

$$S \leftarrow S + \log p_i, \quad A \leftarrow A + c \log p_i.$$

Replace the available scores in these chains.

3. If $S > B_m$, return the retained vectors and best value. Otherwise evaluate $A - \gamma - \log \log S$. A larger value replaces the retained vectors; an equal value adds the current vector. Return to step 2.

Each update processes a complete score block, so every vector evaluated is a prefix. Conversely, every prefix up to the stopping point is evaluated. Theorem 2.2 places every maximizer among these prefixes and excludes all later ones. This proves correctness. The same theorem gives the strict threshold $1/(\log N \log \log N)$ for each returned integer N .

The search is finite because every increment adds at least $\log 2$ to S . There are at most m increments in a tie block, since scores decrease strictly along each prime chain. Thus the number K of increments processed, including the first block beyond the bound, satisfies

$$K \leq \left\lfloor \frac{B_m - \theta_m}{\log 2} \right\rfloor + m.$$

A linear scan of the available scores uses $O(mK)$ comparisons. A heap uses $O(m + K \log m)$ comparisons for score selection and updates. These counts are in the stated comparison model; they exclude prime generation, numerical precision, and the storage of the returned vectors. The score queue itself has m entries. Briggs's CA algorithm uses the same exponent-change events and compresses the long tail of exponents equal to one [4, Section 3.1].

A decreasing tail bound. The best value already found can give an earlier stopping point. Put

$$Q_m = \prod_{i=1}^m \frac{p_i}{p_i - 1}, \quad R(n) = \frac{\sigma(n)}{n \log \log n}.$$

Every integer on the prescribed support with log-size at least S satisfies

$$R(n) < \frac{Q_m}{\log S}.$$

Once this upper bound is below the best value of R found, the remaining tail is excluded. The small-support computation in Appendix A uses this stopping rule.

Interval implementation. The verification supplement represents scores and objective values by Arb enclosures [8]. A comparison is certified when the enclosures establish the required strict inequality. An unresolved comparison terminates the run without a certificate. The mathematical algorithm groups equal scores and retains equal objective values; the completed interval runs have strict comparisons.

The supplement computes the small slices and the fixed slices $6 \leq m \leq 30$ needed for the bounded-support transition. Its saved records include the maximizing exponents, the objective enclosure, and a positive stopping margin. Some of the prefix counts are

m	prefixes evaluated	position of the maximum
6	28	5
10	87	9
20	343	14
29	687	17
30	731	17

Here the primorial is the first prefix. For example, at $m = 6$ the maximizing exponents are $(4, 2, 1, 1, 1, 1)$. The scan continues past the maximum until the stopping inequality is certified. No assumption of unimodality is used.

For bounded support, Section 4.2 makes the first-exponent increments optional and adds the finite boundary created by $n \geq 5041$. The same Euler-product bound terminates the remaining prefix scan once a feasible integer has been found.

4 Small and bounded supports

4.1 Small supports

For $m \leq 5$ the product of the first m primes is smaller than 5041, so the replacement in Proposition 2.1 can leave the range $n \geq 5041$, and the support need not reduce to the first m primes. A finite verification settles these slices.

Proposition 4.1 (Small slices). *For $1 \leq m \leq 5$, the supremum defining $g_{\mathbf{Z}}(m)$ is a maximum, attained uniquely at*

$$2^{13}, \quad 2^6 3^4, \quad 2^5 3^2 5^2, \quad 2^5 3^2 5 \cdot 7, \quad 2^4 3^2 5 \cdot 7 \cdot 11,$$

respectively. Each maximum is negative, and $g_{\mathbf{Z}}(4) = \log G(10080) = -0.0142\dots$ exceeds the other four.

Proof. The local factor of $\sigma(n)/n$ at a prime p is smaller than $p/(p-1)$, and $p/(p-1)$ decreases in p , so $\omega(n) = m$ gives $\sigma(n)/n < \prod_{i=1}^m p_i/(p_i-1)$. Hence

$$\log G(n) < \sum_{i=1}^m \log \frac{p_i}{p_i-1} - \gamma - \log \log \log n.$$

The bound decreases to $-\infty$, so the supremum is attained. The boundary reduction and prefix computation in Appendix A give the stated unique maxima. $\square$

Robin's inequality therefore holds on every slice with $m \leq 5$. The values are not monotone in m , since $g_{\mathbf{Z}}(5) = \log G(55440) = -0.0168 \dots$ falls below the $m = 4$ value.

4.2 At most m prime factors

When $\omega(n) \leq m$, the support is chosen together with the exponents. Define

$$g_{\mathbf{Z}}^{\leq}(m) = \sup_{\substack{n \geq 5041 \\ \omega(n) \leq m}} \log G(n).$$

The CA marginal ordering restricted to the first m primes reduces this problem to a sequence of prefixes and a finite range near the cutoff 5041.

For $p \in \{p_1, \dots, p_m\}$, order all increments $F(p, k)$, $k \geq 1$, by decreasing score, processing equal scores as one block. Starting from 1, let n_j be the integer obtained after the first j blocks. Every prefix is a valid exponent vector with at most m positive exponents. Direct score comparison gives the first score-ordered prefixes

$$2, 6, 12, 60, 120, 360, 2520, 5040, 55440,$$

which agree with Nicolas's CA tables through 2520 and at 5040 and 55440 [13, Tables 1–2]. Thus 55440 is the first score-ordered prefix above Robin's lower cutoff.

Theorem 4.2 (Bounded-support characterization). *For every $m \geq 6$,*

$$g_{\mathbf{Z}}^{\leq}(m) = \max \left\{ \max_{\substack{5041 \leq n < 55440 \\ \omega(n) \leq m}} \log G(n), \max_{\substack{j \geq 1 \\ n_j \geq 55440}} \log G(n_j) \right\}. \quad (4.1)$$

Both maxima are attained. Every maximizing integer at least 55440 is a score-ordered prefix.

Proof. We first reduce an arbitrary support to an initial prime segment. Suppose

$$n = \prod_{i=1}^k q_i^{a_i}, \quad q_1 < \cdots < q_k, \quad k \leq m.$$

Replacing q_i by p_i decreases n and strictly increases $\sigma(n)/n$, unless the support was already initial. If the resulting integer is below 5041, multiply it by the smallest power of 2 that reaches 5041. The result is below 10082, preserves the support size, and further increases the divisor ratio. For $n \geq 55440$, this produces a better feasible integer, either in the finite range in (4.1) or on an initial support.

An integer on an initial support is a finite selection from the ordered increment list. Put $z = \log n$, and choose consecutive prefix integers with

$$\log n_j \leq z \leq \log n_{j+1}.$$

The fractional-knapsack inequality bounds $\log(\sigma(n)/n)$ by the chord joining the corresponding prefix values. This chord is affine in z , while $-\log \log z$ is strictly convex. One of its endpoints therefore has objective at least $\log G(n)$. If $z = \log 55440$, then $n = 55440$, which is already a prefix. If $z > \log 55440$, both adjacent endpoints are feasible. Equality for a maximizing integer then requires equality in the knapsack bound. If the selection cuts a tie block, its point lies in the interior of the affine block segment, where strict convexity makes one endpoint strictly better. At an endpoint, equality in the knapsack bound forces every earlier block to be selected and every later block to be unselected. The total log-size then forces the intervening block to be either complete or absent. Thus every maximizing integer in this range is a score-ordered prefix.

The first maximum in (4.1) is finite. Along the prefixes, the total possible increase in the log divisor ratio is bounded by

$$-\sum_{i=1}^m \log \left(1 - \frac{1}{p_i} \right),$$

whereas $\log \log \log n_j$ tends to infinity. Hence $\log G(n_j)$ tends to minus infinity, and the second maximum is also attained. $\square$

Corollary 4.3 (Unsaturated support). *Let $n > 55440$ maximize $g_{\mathbf{Z}}^{\leq}(m)$, and suppose $k = \omega(n) < m$. Then n is the unique colossally abundant number with parameter*

$$\varepsilon = \frac{1}{\log n \log \log n}.$$

It is also superabundant at its own size, and

$$p_k < \log n < p_{k+1} + 1, \quad k \in \{\pi(\log n) - 1, \pi(\log n)\}.$$

Proof. Let the last selected block have score c_- , and let the next block have score c_+ . On the two adjacent chord intervals, the one-sided derivatives of the interpolated objective at $\log n$ are

$$c_- - \varepsilon > 0, \quad c_+ - \varepsilon < 0.$$

The weak inequalities follow from endpoint maximality, and equality is excluded by strict convexity on the adjacent interval. Thus every selected score exceeds ε , while every eligible unselected score is smaller.

Because $k < m$, the first-exponent increment for p_{k+1} is eligible and unselected. Monotonicity of F extends the strict separation to every larger prime and to all its exponent increments. Equation (2.2) then shows that n uniquely maximizes

$$\frac{\sigma(N)}{N^{1+\varepsilon}}$$

over all positive integers, which is the CA characterization. The same inequality shows that $\sigma(N)/N \leq \sigma(n)/n$ whenever $N \leq n$, so n is SA.

The selected first-exponent increment at p_k and the unselected one at p_{k+1} give

$$F(p_k, 1) > \frac{1}{\log n \log \log n} > F(p_{k+1}, 1).$$

Using

$$\frac{1}{p+1} < \log \left(1 + \frac{1}{p} \right) < \frac{1}{p}$$

and the monotonicity of $x \log x$ gives the endpoint bounds. The statement about k follows. $\square$

We now compare the two maxima in (4.1).

Proposition 4.4 (Eventual prefix dominance). *For every $m \geq 6$, the first maximum in (4.1) equals $\log G(10080) = -0.0142\dots$. For $6 \leq m \leq 29$, it is the larger one, $g_{\mathbf{Z}}^{\leq}(m) = \log G(10080)$, and the unique maximizer is $10080 = 2^5 3^2 5 \cdot 7$. For $m \geq 30$, every maximizer of $g_{\mathbf{Z}}^{\leq}(m)$ is a score-ordered prefix above 55440.*

Proof. Every integer below 55440 has at most six distinct prime factors. Proposition 4.1 gives the maxima on slices 1 through 5, and the prefix

computation of Theorem 2.2 gives $g_{\mathbf{Z}}(6) < \log G(10080)$. Hence the first maximum is attained uniquely at 10080 for every $m \geq 6$.

A score-ordered prefix $n_j \geq 55440$ with k positive exponents lies in the slice $\omega(n) = k$, so its objective is at most $g_{\mathbf{Z}}(k)$, with equality only at a slice maximizer. For $k \leq 5$, Proposition 4.1 gives $g_{\mathbf{Z}}(k) \leq g_{\mathbf{Z}}(4) = \log G(10080)$, with equality only at $k = 4$, whose maximizer 10080 lies below 55440, so the prefix objective is strictly smaller. For $6 \leq k \leq 29$, a verification by the prefix computation of Theorem 2.2 gives $g_{\mathbf{Z}}(k) < \log G(10080)$. The tightest case is $g_{\mathbf{Z}}(29) = -0.0144\dots$, smaller by $0.00012\dots$. Hence the second maximum stays strictly below the first for $6 \leq m \leq 29$, which proves the middle claim.

The same computation gives $g_{\mathbf{Z}}(30) = -0.0135\dots > \log G(10080)$. For $m \geq 30$ the slice $\omega(n) = 30$ is feasible, so $g_{\mathbf{Z}}^{\leq}(m) > \log G(10080)$, no integer in the finite range attains the maximum, and Theorem 4.2 identifies every maximizer as a score-ordered prefix. $\square$

For a maximizing prefix above 55440, the threshold has two interpretations. If $\omega(n) = m$, the CA and size-constrained maximization statements hold over integers whose prime divisors are at most p_m . If $\omega(n) < m$, strict separation extends to all primes, so the optimizer is an unrestricted CA number and hence SA.

5 The continuous optimizer and proximity

Fix $m \geq 6$, with support $(p_1, \dots, p_m)$. For $r \in [1, \infty)^m$, we write

$$S(r) = \sum_{i=1}^m r_i \log p_i.$$

For $r \in \mathbf{Z}_{++}^m$, we have

$$\frac{\sigma(\prod_i p_i^{r_i})}{\prod_i p_i^{r_i}} = \prod_{i=1}^m \frac{p_i - p_i^{-r_i}}{p_i - 1},$$

so the real-exponent relaxation is obtained by using the expression on the right for every $r \in [1, \infty)^m$. Its value is

$$g_{\mathbf{R}}(m) = \sup_{r \in [1, \infty)^m} \left(\sum_{i=1}^m \log(p_i - p_i^{-r_i}) - \gamma - \sum_{i=1}^m \log(p_i - 1) - \log \log S(r) \right). \quad (5.1)$$

This objective agrees with $\log G(\prod_i p_i^{r_i})$ at every positive integer exponent vector. Since the integer exponent vectors form a subset of its feasible set,

$$g_{\mathbf{R}}(m) \geq g_{\mathbf{Z}}(m).$$

The integrality gap is

$$\Delta(m) = g_{\mathbf{R}}(m) - g_{\mathbf{Z}}(m).$$

We refer to $g_{\mathbf{R}}(m)$ as the relaxed excess and to $-g_{\mathbf{Z}}(m)$ as the fixed-slice Robin margin. A positive relaxed excess means that the real-exponent relaxation violates Robin's bound. A positive Robin margin means that every integer in the slice satisfies the bound.

For every $m \geq 6$,

$$-g_{\mathbf{Z}}(m) = \Delta(m) - g_{\mathbf{R}}(m).$$

We estimate the two terms separately. The gap comes from rounding the exponents, and its asymptotic is unconditional. The relaxed value is approximated by a normalized Mertens error, whose behavior depends on the zeros of the zeta function.

5.1 The relaxed optimizer

For $T > 0$, set

$$s_m(T) = \sum_{i=1}^m \max\{\log p_i, \log(1+T) - \log p_i\}.$$

Proposition 5.1 (Relaxed optimizer). *The relaxed problem has a unique maximizer r^* . There is a unique solution $T^* > 0$ of*

$$T = s_m(T) \log s_m(T),$$

and

$$r_i^* = \max\left\{1, \frac{\log(1+T^*)}{\log p_i} - 1\right\}. \quad (5.2)$$

In particular,

$$S^* := S(r^*) = s_m(T^*), \quad T^* = S^* \log S^*.$$

Proof. The local logarithmic terms in (5.1) are bounded above, whereas $\log \log S(r)$ tends to infinity whenever a coordinate of r does. A maximum therefore exists. At any maximum, put $T = S(r) \log S(r)$. Since

$$\frac{d}{du} \log(p_i - p_i^{-u}) = \frac{\log p_i}{p_i^{u+1} - 1},$$

the one-sided first-order conditions give

$$p_i^{r_i+1} - 1 = T \quad \text{if } r_i > 1, \quad p_i^2 - 1 \geq T \quad \text{if } r_i = 1.$$

These conditions are equivalent to (5.2), and the corresponding log-size is $s_m(T)$.

It remains to show that the scalar equation has one solution. Set

$$h(T) = T - s_m(T) \log s_m(T).$$

We have $h(0) = -\theta(p_m) \log \theta(p_m) < 0$, while $s_m(T) \leq \theta(p_m) + m \log(1+T)$, so $h(T)$ tends to infinity. Every root satisfies $T \geq \theta(p_m) \log \theta(p_m)$.

Away from the switching points, let k be the number of primes with $p_i^2 < 1+T$. If $k = 0$, then $h'(T) = 1$. Otherwise,

$$h'(T) = 1 - \frac{k(1 + \log s_m(T))}{1 + T}.$$

The k th prime is at least $k+1$, so $1+T > (k+1)^2$. Also

$$s_m(T) = \theta(p_m) + \sum_{p_i^2 < 1+T} (\log(1+T) - 2 \log p_i) < 2(1+T).$$

Indeed, for $T \geq \theta(p_m) \log \theta(p_m)$, we have $\theta(p_m) < 1+T$, while $k < \sqrt{1+T}$ and $\log(1+T) < \sqrt{1+T}$. The elementary inequality

$$k(1 + \log(2u)) < u, \quad u > (k+1)^2,$$

now gives $h'(T) > 0$. At the left endpoint the inequality reduces to

$$k+1 + \frac{1}{k} - \log 2 - 2 \log(k+1) > 0.$$

This is immediate for $k = 1, 2$ and increasing for $k \geq 2$. Continuity across the switching points proves uniqueness of the root. Every global maximizer satisfies the first-order conditions, so the vector given by (5.2) at this root is the unique maximizer. $\square$

5.2 Asymptotic size

Proposition 5.2 (Relaxed scale). *As m tends to infinity,*

$$S^* \sim \theta(p_m) \sim p_m, \quad T^* \sim p_m \log p_m,$$

and

$$S^* - \theta(p_m) = O\left(\sqrt{\frac{p_m}{\log p_m}}\right). \quad (5.3)$$

Proof. Put $y = \sqrt{1 + T^*}$. We first show that $y \leq p_m$ for all sufficiently large m . Otherwise every coordinate is interior, so

$$S^* = 2m \log y - \theta(p_m) \leq 2m \log y.$$

Since $y^2 - 1 = S^* \log S^* < 2S^* \log y$, this would imply $y/\log y \ll \sqrt{m}$. But $y > p_m$ and the prime number theorem give $y/\log y > p_m/\log p_m \sim m$, a contradiction.

Splitting the response (5.2) at $p = y$ gives

$$\begin{aligned} S^* - \theta(p_m) &= 2(\pi(y) \log y - \theta(y)) \\ &= 2 \int_2^y \frac{\pi(u)}{u} du = O\left(\frac{y}{\log y}\right). \end{aligned}$$

Because $y^2 - 1 = S^* \log S^*$, the last bound is $o(S^*)$. It follows that $S^* \sim \theta(p_m) \sim p_m$, and hence $T^* \sim p_m \log p_m$. Substitution in the same bound proves (5.3). $\square$

Proposition 5.3 (Integer scale). *For every integer maximizer N_m^* ,*

$$\log N_m^* - \theta(p_m) = (\sqrt{2} + o(1)) \sqrt{\frac{T^*}{\log T^*}}, \quad (5.4)$$

and

$$1 - T^* \varepsilon_m = (\sqrt{2} + o(1)) \sqrt{\frac{\log T^*}{T^*}}. \quad (5.5)$$

In particular, $T^* \varepsilon_m < 1$ for all sufficiently large m .

Proof. Put $L_m = \log N_m^*$. Strict separation in Theorem 2.2 gives the exact identity

$$L_m - \theta(p_m) = \sum_{p \leq p_m} \log p \# \{k \geq 2 \mid F(p, k) > \varepsilon_m\}.$$

The bound $F(p, k) \leq 1/(p^k - 1)$ first gives

$$L_m - \theta(p_m) = O\left(\sqrt{L_m \log L_m}\right).$$

Since $L_m = \theta(p_m) + O(\sqrt{L_m \log L_m})$, we have $L_m \sim \theta(p_m) \sim p_m$. Hence

$$L_m \log L_m \sim T^*.$$

We now estimate the total log-size of the selected optional increments. Put

$$U = L_m \log L_m.$$

The first optional increment satisfies

$$F(p, 2) = \frac{1 + o(1)}{p^2 \log p}.$$

For every fixed $\delta > 0$ and every sufficiently large p , this gives the uniform bounds

$$\frac{1 - \delta}{p^2 \log p} \leq F(p, 2) \leq \frac{1 + \delta}{p^2 \log p}.$$

The positive solution of $x^2 \log x = U$ is

$$(\sqrt{2} + o(1)) \sqrt{\frac{U}{\log U}}.$$

Comparison with the positive solutions of $x^2 \log x = (1 \pm \delta)U$, followed by the prime number theorem and then $\delta \rightarrow 0$, gives

$$\sum_{\substack{p \leq p_m \\ F(p, 2) > 1/U}} \log p = (\sqrt{2} + o(1)) \sqrt{\frac{U}{\log U}}.$$

The support cutoff is inactive because

$$p_m \asymp \frac{T^*}{\log T^*} \gg \sqrt{\frac{U}{\log U}}.$$

For $k \geq 3$, the bound $F(p, k) \leq 1/(p^k - 1)$ gives total log weight

$$\sum_{3 \leq k \leq \log_2(1+U)} \theta\left((1+U)^{1/k}\right) = O(U^{1/3} + U^{1/4} \log U),$$

which is lower order. Since $U \sim T^\star$, the identity for $L_m - \theta(p_m)$ proves (5.4).

By (5.3),

$$S^\star - \theta(p_m) = o\left(\sqrt{\frac{T^\star}{\log T^\star}}\right).$$

Together with (5.4), this gives

$$L_m - S^\star = (\sqrt{2} + o(1))\sqrt{\frac{T^\star}{\log T^\star}}.$$

Taylor expansion of $u \log u$ now gives

$$\begin{aligned} L_m \log L_m - T^\star &= (1 + \log S^\star)(L_m - S^\star) + O\left(\frac{(L_m - S^\star)^2}{S^\star}\right) \\ &= (\sqrt{2} + o(1))\sqrt{T^\star \log T^\star}. \end{aligned}$$

Taking reciprocals proves (5.5). □

5.3 One-step proximity

Fix an integer maximizer

$$N_m^\star = \prod_{i=1}^m p_i^{a_i},$$

and put $Y = \log(1 + T^\star)$. For all sufficiently large m , its exponents are adjacent to the relaxed exponents.

Proposition 5.4 (One-step proximity). *For every sufficiently large m ,*

$$a_i \in \left\{ \left\lfloor \frac{Y}{\log p_i} - 1 \right\rfloor, \left\lceil \frac{Y}{\log p_i} - 1 \right\rceil \right\}$$

when $p_i < e^{Y/2}$. For $p_i \geq e^{Y/2}$, we have $a_i = 1$.

Proof. Fix a prime below $e^{Y/2}$, and put

$$\ell = \left\lfloor \frac{Y}{\log p} - 1 \right\rfloor.$$

The relaxed first-order equation is

$$\frac{1}{p^{Y/\log p} - 1} = \frac{1}{T^\star}.$$

If $\ell \geq 2$, the increment to exponent ℓ lies below the relaxed coordinate. Its score is greater than $1/T^*$, and hence greater than ε_m by Proposition 5.3. It is selected. When $\ell = 1$, the support constraint already gives $a_i \geq \ell$.

We next rule out the increment to exponent $\ell + 2$. Write

$$u = \frac{Y}{\log p} - 1 + v$$

on the corresponding unit interval. Since $p^{Y/\log p} = 1 + T^*$,

$$\frac{T^*}{p^{u+1} - 1} \leq p^{-v}.$$

The interval begins at some $v \in (0, 1]$, so

$$T^* F(p, \ell + 2) \leq \int_0^1 p^{-v} dv = \frac{1 - p^{-1}}{\log p} \leq \frac{1}{2 \log 2} < 1.$$

Since $T^* \varepsilon_m$ tends to one, this increment and all later ones are unselected. Thus $\ell \leq a_i \leq \ell + 1$.

If $Y/\log p - 1$ is an integer, the same estimate applied to the next increment shows that it is unselected. This proves the stated floor–ceiling alternative. If $p_i \geq e^{Y/2}$, apply the same estimate to $F(p_i, 2)$. The corresponding unit interval begins at a nonnegative value of v , so this increment is unselected and $a_i = 1$. $\square$

6 Score rounding and the integrality gap

Fix an integer maximizer $N_m^* = \prod_i p_i^{a_i}$, and put $Y = \log(1 + T^*)$. We now evaluate the loss from integral exponents.

Theorem 6.1 (Integrality-gap asymptotic). *The integrality gap satisfies*

$$\Delta(m) = \frac{2\sqrt{2} + o(1)}{\sqrt{p_m} \log p_m}.$$

To compare the optimizers coordinatewise, define an auxiliary integer exponent at the relaxed threshold $1/T^*$. The threshold separation in (5.5) controls its difference from the maximizing exponent, which uses ε_m . For $Y > 2 \log 2$ and $\log 2 \leq x \leq Y/2$, put

$$a_Y(x) = \begin{cases} \lfloor Y/x \rfloor, & F(e^x, \lfloor Y/x \rfloor) > (e^Y - 1)^{-1}, \\ \lfloor Y/x \rfloor - 1, & F(e^x, \lfloor Y/x \rfloor) \leq (e^Y - 1)^{-1}. \end{cases} \quad (6.1)$$

This is one of the two integers adjacent to $Y/x - 1$. It rounds upward exactly when the increment score exceeds the relaxed threshold. Define its local loss by

$$\kappa(x, Y) = \log \frac{1 - e^{-Y}}{1 - e^{-x(a_Y(x)+1)}} + \frac{x(a_Y(x) + 1) - Y}{e^Y - 1}, \quad (6.2)$$

and define

$$J(Y) = \int_{\log 2}^{Y/2} \kappa(x, Y) \frac{e^x}{x} dx. \quad (6.3)$$

Thus κ is the loss in the local divisor factor after the first-order log-size change has been removed.

Proof of Theorem 6.1. Rounding reduction. For $p_i < e^{Y/2}$, let $b_i = a_Y(\log p_i)$. Subtracting the two objective values and linearizing the size penalty at S^* gives

$$\begin{aligned} \Delta(m) &= \sum_{p_i < e^{Y/2}} \left[\log \frac{p_i - p_i^{-r_i^*}}{p_i - p_i^{-a_i}} - \frac{(r_i^* - a_i) \log p_i}{T^*} \right] \\ &\quad + O\left(\frac{\log T^*}{(T^*)^2} (L_m - S^*)^2\right). \end{aligned}$$

Proposition 5.4 gives

$$|L_m - S^*| \leq \theta(e^{Y/2}) = O(\sqrt{T^*}),$$

so the displayed error is lower order on the required scale.

Both a_i and b_i are adjacent to r_i^* . If they differ, the intervening score lies between ε_m and $1/T^*$. Hence replacing a_i by b_i in the sum changes it by at most

$$\theta(e^{Y/2}) \left(\frac{1}{T^*} - \varepsilon_m \right) = O\left(\frac{\sqrt{\log T^*}}{T^*}\right),$$

where we used (5.5). This is also lower order. The bracket with b_i is exactly $\kappa(\log p_i, Y)$, by (6.2). We have proved

$$\sum_{p < e^{Y/2}} \kappa(\log p, Y) = \Delta(m) + o\left(\frac{1}{\sqrt{T^* \log T^*}}\right).$$

Prime-sum approximation. Put $q = e^{-Y}$ and $\lambda = (e^Y - 1)^{-1}$. On an interval where $k = \lfloor Y/x \rfloor$ is constant, put $s = Y - kx$. Formula (6.2) becomes

$$\kappa(x, Y) = \begin{cases} \log(1 - q) - \log(1 - qe^s) - s\lambda, & F(e^x, k) \leq \lambda, \\ \log(1 - q) - \log(1 - qe^{s-x}) + (x - s)\lambda, & F(e^x, k) > \lambda. \end{cases} \quad (6.4)$$

The difference between the branches is

$$x(\lambda - F(e^x, k)),$$

so they agree at a switch.

On the first branch,

$$\begin{aligned} xF(e^x, k) &= \log(1 - qe^{s-x}) - \log(1 - qe^s) \\ &\geq qe^s(1 - e^{-x}) \geq \frac{1}{2}qe^s. \end{aligned}$$

The branch condition therefore gives $qe^s \ll xq$. On the second branch, $qe^{s-x} \leq q$. It follows from (6.4) that

$$|\kappa(x, Y)| \ll Ye^{-Y}.$$

Where k is constant, $s' = -k$, and differentiation gives

$$|\partial_x \kappa(x, Y)| \ll Ye^{-Y}$$

on every smooth piece. There are $O(Y)$ intervals on which k is constant. The score

$$F(e^x, k) = \int_k^{k+1} \frac{du}{e^{xu} - 1}$$

is (2.3) after shifting the integration variable. It decreases strictly with x , so each interval has at most one branch switch. The smooth pieces have total length $O(Y)$, and the jumps between successive intervals are $O(Ye^{-Y})$. Consequently, the total variation satisfies

$$\text{TV}_{[\log 2, Y/2]} \kappa(\cdot, Y) \ll Y^2 e^{-Y}.$$

At $x = Y/2$, the score-rounded exponent is one and $\kappa(Y/2, Y) = 0$.

The de la Vallée Poussin estimate [6] gives

$$\pi(e^x) - \text{li}(e^x) = O\left(\frac{e^x}{x} e^{-c\sqrt{x}}\right).$$

Stieltjes integration by parts, using the variation bound and the vanishing upper endpoint, now gives

$$\begin{aligned} \sum_{p < e^{Y/2}} \kappa(\log p, Y) - \int_{\log 2}^{Y/2} \kappa(x, Y) \frac{e^x}{x} dx \\ = O\left(Ye^{-Y/2} e^{-c'\sqrt{Y}}\right) + O(Ye^{-Y}). \end{aligned}$$

This is lower order than the required scale. The choice of endpoint values at $x = Y/k$ changes the expression by at most $O(Y^2 e^{-Y})$. It follows that

$$\sum_{p < e^{Y/2}} \kappa(\log p, Y) = J(Y) + o\left(\frac{1}{\sqrt{(e^Y - 1)Y}}\right).$$

Second-exponent range. The branch formulas give the sharper bound

$$|\kappa(x, Y)| \ll q(1 + x).$$

Therefore

$$\int_{\log 2}^{Y/3} |\kappa(x, Y)| \frac{e^x}{x} dx = O(e^{-2Y/3}),$$

which is lower order. Only $Y/3 < x \leq Y/2$, where $\lfloor Y/x \rfloor = 2$, contributes at first order.

For all sufficiently large Y , let $x_\star$ be the solution in $(Y/3, Y/2)$ of

$$F(e^{x_\star}, 2) = \frac{1}{e^Y - 1}. \quad (6.5)$$

On this interval put

$$u = Y - 2x, \quad x = \frac{Y - u}{2}.$$

The score in (6.5) increases with u . It is below $(e^Y - 1)^{-1}$ at $u = 0$ and above it near $u = Y/3$, so the switch exists and is unique. Put $u_\star = Y - 2x_\star$. Uniformly on the interval,

$$\begin{aligned} xF(e^x, 2) &= \log(1 - e^{-3x}) - \log(1 - e^{-2x}) \\ &= e^{-Y+u}(1 + O(e^{-x} + e^{-Y+u})). \end{aligned}$$

The switch equation gives

$$e^{u_\star} = x_\star(1 + O(e^{-Y/3})).$$

Together with $2x_\star + u_\star = Y$, this proves

$$2x_\star + \log x_\star = Y + o(1).$$

On the two sides of the switch, (6.4) gives

$$\kappa(x, Y) = \begin{cases} q(e^u - 1 - u) + O(q^2 e^{2u}), & 0 \leq u \leq u_\star, \\ q(x - u - 1 + e^{-(x-u)}) + O(Yq^2), & u_\star \leq u < Y/3. \end{cases}$$

Since $dx = -du/2$ and $e^x = e^{Y/2}e^{-u/2}$, the contribution from each branch is $qe^{Y/2}/2$ times its corresponding integral below. For the first branch,

$$\int_0^{u_\star} \frac{e^{-u/2}}{x} (e^u - 1 - u) du = \frac{2 + o(1)}{\sqrt{x_\star}}.$$

Indeed, the terms $1 + u$ contribute $O(x_\star^{-1})$. In the remaining term, set $t = u_\star - u$, use $x = x_\star + t/2$ and $e^{u_\star} \sim x_\star$, and apply dominated convergence.

For the second branch, set $t = u - u_\star$. Then

$$x = x_\star - \frac{t}{2}, \quad x - u = x_\star - u_\star - \frac{3t}{2}.$$

Since

$$0 \leq x - u - 1 + e^{-(x-u)} \leq x - u \leq x,$$

the second integral equals

$$e^{-u_\star/2} \int_0^{Y/3-u_\star} e^{-t/2} \frac{x - u - 1 + e^{-(x-u)}}{x} dt.$$

The ratio is bounded by one and tends to one for every fixed t . Since $e^{-u_\star/2} \sim x_\star^{-1/2}$, dominated convergence gives

$$\int_{u_\star}^{Y/3} \frac{e^{-u/2}}{x} (x - u - 1 + e^{-(x-u)}) du = \frac{2 + o(1)}{\sqrt{x_\star}}.$$

After multiplication by the Jacobian factor $e^{Y/2}/2$, the error $O(q^2 e^{2u})$ on the first branch contributes

$$O(e^{-3Y/2} x_\star^{1/2}),$$

while the error $O(Yq^2)$ on the second branch contributes

$$O(e^{-3Y/2} x_\star^{-1/2}).$$

Both are $o(e^{-Y/2} x_\star^{-1/2})$, the scale of the main term. Combining the two branches yields

$$J(Y) = e^{-Y/2} \left(\frac{2}{\sqrt{x_\star}} + o(x_\star^{-1/2}) \right).$$

Finally, $x_\star \sim Y/2$ and

$$e^{-Y/2} = \frac{\sqrt{1 - e^{-Y}}}{\sqrt{e^Y - 1}}.$$

Consequently,

$$J(Y) = \frac{1}{\sqrt{e^Y - 1}} \left(\frac{2}{\sqrt{x_\star}} + o(x_\star^{-1/2}) \right) = \frac{2\sqrt{2} + o(1)}{\sqrt{(e^Y - 1)Y}}.$$

The two sides of the switch contribute equally at first order. Since $e^Y - 1 = T^\star$, the rounding reduction and prime-sum approximation give

$$\Delta(m) = \frac{2\sqrt{2} + o(1)}{\sqrt{T^\star Y}}.$$

Since $Y \sim \log T^\star$ and $T^\star \sim p_m \log p_m$, the denominator is asymptotic to $\sqrt{p_m \log p_m}$. $\square$

6.1 CA exponent thresholds and the Axler–Nicolas comparison

The classical CA exponent thresholds give a second description of the integer optimizer and another derivation of the gap asymptotic.

Define the integer exponent thresholds by

$$F(\xi, 1) = \varepsilon_m, \quad F(\xi_k, k) = \varepsilon_m.$$

Nicolas [13, Proposition 4.7] proves

$$\xi^{1/k} \leq \xi_k \leq (k\xi)^{1/k}, \quad \xi_2 \sim \sqrt{2\xi}.$$

Since

$$F(\xi, 1) \sim \frac{1}{\xi \log \xi},$$

Proposition 5.3 gives $\xi \sim L_m \sim p_m$. The exact threshold rule is

$$L_m - \theta(p_m) = \sum_{k \geq 2} \theta(\min\{p_m, \xi_k\}).$$

Nicolas's bounds give $\xi_k < p_m$ for every $k \geq 2$ and all sufficiently large m . Hence

$$L_m - \theta(p_m) = \sum_{k \geq 2} \theta(\xi_k) = \theta(\xi_2) + O(\xi^{1/3}) = \sqrt{2p_m} + o(\sqrt{p_m}).$$

The middle estimate follows from Nicolas's bounds for ξ_k , the prime number theorem, and the fact that only $O(\log \xi)$ thresholds are at least two.

The rounding threshold e^{x^*} is defined by

$$F(e^{x^*}, 2) = \frac{1}{T^*},$$

whereas ξ_2 uses ε_m . Equation (5.5) shows that

$$e^{x^*} \sim \xi_2 \sim \sqrt{2p_m}.$$

Factoring the integer objective as in the next section gives the following correction to the Mertens term:

$$\sum_{i=1}^m \log(1 - p_i^{-(a_i+1)}) - (\log \log L_m - \log \log \theta(p_m)).$$

For $\xi_2 < p \leq p_m$, the exponent of p is one. The prime-square tail estimate [13, equations (2.10)–(2.11)] gives

$$\sum_{\xi_2 < p \leq p_m} \log(1 - p^{-2}) = -\frac{1}{\xi_2 \log \xi_2} + o\left(\frac{1}{\sqrt{p_m} \log p_m}\right) = -\frac{\sqrt{2} + o(1)}{\sqrt{p_m} \log p_m}.$$

The remaining Euler factors are lower order. Primes with exponent two lie above ξ_3 , so their contribution is

$$O\left(\sum_{p > \xi_3} p^{-3}\right) = O(\xi^{-2/3}).$$

If $a_i \geq 3$, the next increment is unselected. Hence $p_i > \xi_{a_i+1}$, and Nicolas's lower bound gives $p_i^{a_i+1} > \xi$. There are $O(\xi_3) = O(\xi^{1/3})$ such primes, so their total contribution is also $O(\xi^{-2/3})$. Consequently,

$$\sum_{i=1}^m \log(1 - p_i^{-(a_i+1)}) = -\frac{\sqrt{2} + o(1)}{\sqrt{p_m} \log p_m}.$$

Taylor expansion and the formula for $L_m - \theta(p_m)$ similarly give

$$\log \log L_m - \log \log \theta(p_m) = \frac{\sqrt{2} + o(1)}{\sqrt{p_m} \log p_m}.$$

Together with the relaxed-value estimate (7.2), the two displays give another proof of Theorem 6.1. This proof separates the correction into an Euler-factor tail and a log-size change. The rounding proof separates it into local

losses from downward and upward rounding and shows that their leading contributions are equal.

Nicolas's expansion of the Axler–Nicolas size-constrained comparison [14, equations (5.3)–(5.5)] contains the corresponding terms

$$\frac{1}{\xi_2 \log \xi_2}, \quad \frac{\xi_2}{\xi \log \xi}.$$

The first is the same leading Euler-factor tail. In Nicolas's comparison, the second term measures the contribution of the additional primes allowed by the common size constraint. In the present problem, it measures the change in $\log \log L_m$ caused, to first order, by the increments from exponent one to exponent two. Since $\xi_2 \sim \sqrt{2\xi}$, each term contributes $\sqrt{2}/(\sqrt{\xi} \log \xi)$.

Since the Axler–Nicolas ratio tends to one, its relative gap and its logarithm have the same leading term. To compare the normalizations, let P be the largest prime such that $\prod_{p \leq P} p \leq X$. Then $\log X \sim \theta(P) \sim P$. Hence the Axler–Nicolas scale satisfies

$$\frac{1}{\sqrt{\log X} \log \log X} \sim \frac{1}{\sqrt{P} \log P},$$

which is the fixed-slice scale when $P = p_m$.

There is a related relaxation interpretation. For every prime p ,

$$\frac{\sigma(p^a)}{p^a} \longrightarrow \frac{1}{1 - p^{-1}} = \frac{p^a}{\varphi(p^a)}$$

as a tends to infinity. The numerator in the Axler–Nicolas ratio uses these limiting factors. Its denominator uses finite integer exponents, and both maxima allow the support to vary subject to a size constraint. The present problem compares finite real and integer exponents on a fixed support. In both cases, the leading correction comes from the transition from exponent one to exponent two.

7 Consequences for Robin's inequality

7.1 The analytic term

Define the normalized Mertens error

$$E_\theta(x) = - \sum_{p \leq x} \log \left(1 - \frac{1}{p} \right) - \gamma - \log \log \theta(x). \quad (7.1)$$

The difference between the relaxed value and this term is asymptotically smaller than the integrality gap.

Relaxed value. We first show that

$$g_{\mathbf{R}}(m) = E_{\theta}(p_m) + O\left(\frac{1}{\sqrt{p_m} \log^{3/2} p_m}\right). \quad (7.2)$$

Put $y = \sqrt{1 + T^*}$. The response (5.2) gives

$$p_i - p_i^{-r_i^*} = \frac{p_i T^*}{1 + T^*}$$

for $p_i \leq y$, while $r_i^* = 1$ for $p_i > y$. Substitution in (5.1) gives

$$\begin{aligned} g_{\mathbf{R}}(m) &= \pi(y) \log \frac{T^*}{1 + T^*} - \sum_{p \leq p_m} \log \left(1 - \frac{1}{p}\right) \\ &\quad + \sum_{y < p \leq p_m} \log \left(1 - \frac{1}{p^2}\right) - \gamma - \log \log S^*. \end{aligned}$$

The prime number theorem and partial summation give

$$\pi(y) \left| \log \frac{T^*}{1 + T^*} \right| + \sum_{p > y} \left| \log \left(1 - \frac{1}{p^2}\right) \right| = O\left(\frac{1}{\sqrt{T^*} \log T^*}\right).$$

Proposition 5.2 and (5.3) also give

$$\log \log S^* - \log \log \theta(p_m) = O\left(\frac{1}{\sqrt{T^*} \log T^*}\right).$$

The result follows from $T^* \sim p_m \log p_m$.

7.2 Under the Riemann hypothesis

Assume the Riemann hypothesis, and define

$$\mathcal{H}(t) = \sum_{\rho} \frac{e^{(\rho-1/2)t}}{\rho(\rho-1)}, \quad Z = 2 + \gamma - \log(4\pi),$$

where the sum runs over the nontrivial zeros of the zeta function, with multiplicity. The series is absolutely convergent because its terms are $O(|\operatorname{Im} \rho|^{-2})$.

Nicolas's integrated explicit formula [12, Theorem 3(b), final formula in its proof], translated to the normalization (7.1), is

$$E_{\theta}(x) = \frac{2 - \mathcal{H}(\log x)}{\sqrt{x} \log x} + O\left(\frac{1}{\sqrt{x} \log^2 x}\right). \quad (7.3)$$

The same proof records the classical identity

$$\sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma - \log(4\pi).$$

On the critical line, $\rho(1-\rho) = |\rho|^2$. Therefore

$$|\mathcal{H}(t)| \leq Z. \quad (7.4)$$

Theorem 7.1 (Relaxed excess under the Riemann hypothesis). *Assume the Riemann hypothesis. Then*

$$\sqrt{p_m} \log p_m g_{\mathbf{R}}(m) = 2 - \mathcal{H}(\log p_m) + O\left(\frac{1}{\sqrt{\log p_m}}\right).$$

In particular,

$$2 - Z + o(1) \leq \sqrt{p_m} \log p_m g_{\mathbf{R}}(m) \leq 2 + Z + o(1),$$

and $g_{\mathbf{R}}(m) > 0$ for all sufficiently large m .

Proof. Apply (7.3) at $x = p_m$, and use (7.2). The error in (7.2), after multiplication by $\sqrt{p_m} \log p_m$, is $O(1/\sqrt{\log p_m})$. The bounds follow from (7.4). The lower one is eventually positive because $2 - Z > 0$. $\square$

7.3 If the Riemann hypothesis fails

We use $f = \Omega_{\pm}(g)$ when f/g has a positive limsup and a negative liminf.

Theorem 7.2 (Oscillations when the hypothesis fails). *Suppose the Riemann hypothesis is false, and let*

$$\Theta = \sup\{\operatorname{Re} \rho \mid \zeta(\rho) = 0\}.$$

For every $0 < \varepsilon < \Theta - 1/2$,

$$g_{\mathbf{R}}(m) = \Omega_{\pm}\left(p_m^{-1/2+\varepsilon}\right).$$

Proof. Nicolas [12, Theorem 3(c)] proves, in the normalization (7.1), that

$$E_{\theta}(x) = \Omega_{\pm}(x^{-b}) \quad (1 - \Theta < b < 1/2).$$

The function E_{θ} is constant between consecutive primes, and the prime number theorem makes the preceding prime asymptotic to x . The two real

sequences in Nicolas's result therefore give two prime subsequences with the same orders and opposite signs.

By (7.2),

$$g_{\mathbf{R}}(m) = E_{\theta}(p_m) + O\left(\frac{1}{\sqrt{p_m} \log^{3/2} p_m}\right).$$

The error is $o(p_m^{-b})$. Taking $b = 1/2 - \varepsilon$ proves the result. $\square$

7.4 The integer margin

The two preceding results, together with $-g_{\mathbf{Z}}(m) = \Delta(m) - g_{\mathbf{R}}(m)$, give the integer margin.

Theorem 7.3 (Fixed-slice Robin margin). *Under the Riemann hypothesis,*

$$-\sqrt{p_m} \log p_m g_{\mathbf{Z}}(m) = 2\sqrt{2} - 2 + \mathcal{H}(\log p_m) + o(1). \quad (7.5)$$

Consequently,

$$g_{\mathbf{R}}(m) > 0 > g_{\mathbf{Z}}(m)$$

for every sufficiently large m .

If the Riemann hypothesis is false, then

$$\begin{aligned} \limsup_{m \rightarrow \infty} -\sqrt{p_m} \log p_m g_{\mathbf{Z}}(m) &= +\infty, \\ \liminf_{m \rightarrow \infty} -\sqrt{p_m} \log p_m g_{\mathbf{Z}}(m) &= -\infty. \end{aligned}$$

Proof. Under the Riemann hypothesis, subtract Theorem 7.1 from the gap formula in Theorem 6.1. This gives (7.5). The relaxed value is eventually positive by Theorem 7.1. The integer value is eventually negative because

$$2\sqrt{2} - 2 + \mathcal{H}(\log p_m) \geq 2\sqrt{2} - 2 - Z > 0.$$

If the hypothesis is false, Theorem 7.2 gives $g_{\mathbf{R}}(m) = \Omega_{\pm}(p_m^{-b})$ for every $1 - \Theta < b < 1/2$. The integrality gap is

$$O\left(\frac{1}{\sqrt{p_m} \log p_m}\right) = o(p_m^{-b}).$$

Since $-g_{\mathbf{Z}} = \Delta - g_{\mathbf{R}}$, the positive and negative subsequences give the stated limits. $\square$

Corollary 7.4 (Eventual fixed- ω criterion). *The Riemann hypothesis is equivalent to the existence of an M such that Robin's inequality holds for every $n \geq 5041$ with $\omega(n) = m$, for every $m \geq M$.*

Proof. If the Riemann hypothesis holds, the theorem gives $g_{\mathbf{Z}}(m) < 0$ for all sufficiently large m , so every integer in each such slice satisfies Robin’s inequality.

If it fails, the theorem gives infinitely many m with $g_{\mathbf{Z}}(m) > 0$. Proposition 2.1 shows that the maximum is attained, so each such slice contains a violation. $\square$

8 Conclusion

For $m \geq 6$, every fixed- ω optimizer is a score-ordered prefix, and the log-size bound gives a finite computation when exact comparisons, including equality, are available. The same ordering treats bounded supports after the first-exponent increments become optional and the lower boundary is included.

The real-exponent problem has a unique optimizer determined by a scalar equation. For all sufficiently large m , the integer exponents are within one step of the relaxed exponents. Approximating the sum of their rounding losses by an integral gives a gap asymptotic to $2\sqrt{2}/(\sqrt{p_m} \log p_m)$, with equal contributions from the two sides of the second-exponent switch. Classical CA exponent thresholds locate this switch asymptotically and explain the common coefficient in the Axler–Nicolas size-constrained comparison. Nicolas’s Mertens formula then gives the eventual fixed- ω criterion.

Effective bounds for the separation of the real and integer thresholds would give a smaller interval of prefixes to examine. Proposition 5.3 estimates this separation asymptotically.

A Small supports and the finite boundary range

The verification supplement¹ contains the computation proving the small-slice maxima and the bounded-support transition. Divisor sums are computed exactly, and comparisons use Arb interval arithmetic [8] through `python-flint` 0.9.0. An unresolved comparison terminates the computation without certifying the claim.

For $1 \leq m \leq 5$, let A_m be the first feasible prefix on the first m primes, with first-exponent increments mandatory. Direct score comparison gives

$$B := \max\{2 \cdot 5041, A_1, \dots, A_5\} = 21600.$$

¹Version 0.2.0: <https://github.com/epicyclics/robin-integrality-gap/tree/v0.2.0/verify>

Every small-slice maximizer is either below B or a prefix. Replacing a noninitial support by the first m primes strictly increases the divisor ratio and decreases the size. If this gives an integer below 5041, multiply by the least power of 2 that makes it feasible. The result is below $2 \cdot 5041 \leq B$. Thus every noninitial-support integer at least B has a feasible improvement, and a maximizer in that range has initial support. Its adjacent prefixes are feasible because $B \geq A_m$. The fractional-knapsack bound and strict convexity used in Theorem 2.2 then force it to be a prefix.

An exact integer sieve checks $[5041, B)$. The remaining candidates are prefixes, scanned until the decreasing Euler-product bound in Proposition 4.1 falls below the best value found. This bounds the entire remaining tail. Strict comparisons among the candidates give uniqueness.

The supplement also describes the exact integer screening used before the interval comparisons.

For $6 \leq m \leq 30$, the prefix scan uses interval score comparisons and continues until the log-size is provably above $\theta(p_m)^{\zeta(2)}$, as required by Theorem 2.2. No assumption of unimodality is used. The boundary maximum in Proposition 4.4 follows from the maxima on slices 1 through 6, since the product of the first seven primes exceeds the boundary endpoint. The first nine bounded-support prefixes are checked in a finite score pool. Monotonicity in (2.3) bounds all omitted primes by $F(101, 1)$ and all omitted exponents by $F(2, 13)$; both are strictly below the ninth selected score.

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